

Towards Crossing-Free Hamiltonian Cycles in Simple Drawings of Complete Graphs

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Abstract

It is a longstanding conjecture that every simple drawing of a complete graph on $n \geq 3$ vertices contains a crossing-free Hamiltonian cycle. We strengthen this conjecture to “there exists a crossing-free Hamiltonian path between each pair of vertices” and show that this stronger conjecture holds for several classes of simple drawings, including strongly c -monotone drawings and cylindrical drawings. As a second main contribution, we give an overview on different classes of simple drawings and investigate inclusion relations between them up to weak isomorphism.

Keywords and phrases crossing-free Hamiltonian cycles and paths, classes of simple drawings

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1 Introduction

In a *drawing* of a graph in the plane (or on the sphere) vertices are represented by distinct points and edges by Jordan arcs connecting the respective points. For convenience we also call those representations vertices and edges. A *simple drawing* is a drawing of a graph where each pair of edges meets in at most one point, which has to be either a proper crossing or a common end-vertex. A fundamental line of research in this area is concerned with finding *crossing-free* sub-drawings in simple drawings of the complete graph K_n on n vertices. These are sub-drawings with pairwise non-crossing edges, also called *plane* sub-drawings. For this task it is sufficient to know all crossings in the drawing, its precise shape is negligible. We call two simple drawings \mathcal{D} and \mathcal{D}' of the same graph *weakly isomorphic* if two edges in \mathcal{D} cross if and only if the corresponding edges in \mathcal{D}' cross. We call them *strongly isomorphic* if there exists a homeomorphism (of the sphere) that maps \mathcal{D} to \mathcal{D}' . Weak isomorphism classes of K_n can be uniquely represented by rotation systems. The *rotation* of a vertex v is the (clockwise) circular order of all edges that are incident to the vertex v , where these edges



are classically identified by their second end-vertex. The collection of the rotations of all vertices in a simple drawing is called its *rotation system*. See Ábrego et al. [1] and Kynčl [28] for more information on rotation systems and see Figure 5 for all non-isomorphic simple drawings of K_5 . In 1988, Nabil Rafla stated the following conjecture in his PhD thesis [32].

► **Conjecture 1.1** (Rafla [32]). *Every simple drawing of the complete graph K_n on $n \geq 3$ vertices contains at least one crossing-free Hamiltonian cycle.*

Related work. Under the assumption that Conjecture 1.1 is true, Rafla enumerated all different simple drawings of K_n for $n \leq 7$ up to weak isomorphism. Since then, Conjecture 1.1 and relaxations of it have attracted considerable attention. Especially, a crossing-free Hamiltonian cycle in a simple drawing \mathcal{D} implies, by removing an arbitrary edge of the cycle, that \mathcal{D} also contains a crossing-free Hamiltonian path. Furthermore, for even n , a crossing-free Hamiltonian path in turn implies that \mathcal{D} contains a plane perfect matching, by disregarding every second edge of the path. However, even the question of the existence of a plane perfect matching in every simple drawing of K_n , for even n , is still open.

In 2003, Pach, Solymosi, and Tóth [31] showed that every simple drawing of K_n contains plane sub-drawings isomorphic to any tree of size $\mathcal{O}(\log(n)^{1/6})$. This immediately implies a lower bound of $\Omega(\log(n)^{1/6})$ for the longest crossing-free path and largest plane matching in every simple drawing of K_n . Subsequently, a lot of progress has been made with regard to plane matchings. Until recently, a lower bound of $\Omega(n^{1/2-\varepsilon})$ for any fixed $\varepsilon > 0$, shown by Ruiz-Vargas [33], was best known. This bound has lately been improved to $\Omega(\sqrt{n})$ in [6], via the introduction and use of generalized twisted drawings; see Section 2 for a definition of generalized twisted drawings and several other drawing classes mentioned in this introduction. In the same paper, and independently also by Suk and Zeng [34], a lower bound of $\Omega(\log(n)/\log(\log(n)))$ for the longest crossing-free path was shown. This is the first improvement for crossing-free paths in almost 20 years since the result from Pach, Solymosi, and Tóth. Furthermore, the authors of [6] obtained the same bound of $\Omega(\log(n)/\log(\log(n)))$ for the longest crossing-free cycle. For a more detailed history on the developments during the last 20 years, we point the interested reader to the references given in [6, 33, 34].

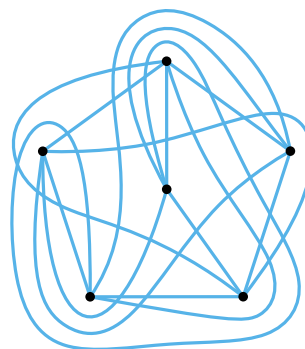
In a similar direction, Pach, Solymosi, and Tóth [31] showed that every simple drawing of K_n contains a sub-drawing of size $\Omega(\log(n)^{1/8})$ that is weakly isomorphic to either a convex straight-line drawing or a so-called twisted drawing. This implies the existence of various plane sub-drawings of the respective size. Recently, Suk and Zeng [34] improved this $\Omega(\log(n)^{1/8})$ bound to $\Omega(\log(n)^{1/4-\varepsilon})$ for any fixed $\varepsilon > 0$. This also improves the above mentioned $\mathcal{O}(\log(n)^{1/6})$ bound for trees accordingly.

Furthermore, Ruiz-Vargas [33] showed that every c -monotone drawing of K_n contains a plane matching of size $\Omega(n^{1-\varepsilon})$ for any fixed $\varepsilon > 0$; so “almost” a perfect matching. In addition, in [6] it was shown that every c -monotone drawing contains a sub-drawing of size $\Omega(\sqrt{n})$ that is weakly isomorphic to either an x -monotone drawing or a generalized twisted drawing, implying that c -monotone drawings of K_n contain a crossing-free path as well as a crossing-free cycle of size $\Omega(\sqrt{n})$.

Concerning crossing-free Hamiltonian cycles, Conjecture 1.1 has been confirmed for all simple drawings on $n \leq 9$ vertices using the rotation system database [1], and Ebenföhrer tested the conjecture on randomly generated realizable rotation systems for up to 30 vertices in his Master’s thesis [17]. Moreover, Arroyo, Richter, and Sunohara [12] proved the existence of a crossing-free Hamiltonian cycle in so-called pseudospherical (or h -convex) drawings of K_n . In a current paper, Bergold, Felsner, M. Reddy, and Scheucher [15] extended this to (generalized) convex drawings. Moreover, they confirmed Conjecture 1.1 for $n = 10$ using a SAT encoding.

Finally, in [6], Conjecture 1.1 was shown to be true for generalized twisted drawings on an odd number of vertices.

On the negative side, in [5, 17] the authors showed that simplicity of the drawings is crucial: They provided a star-simple drawing of K_6 that does not contain any “crossing-free” Hamiltonian cycle (see Figure 1). A *star-simple drawing* is a drawing where incident edges must not cross but non-incident edges are allowed to cross more than once. We remark that in the context of star-simple drawings, edges are actually only considered to be “crossing” when they cross an odd number of times.



■ **Figure 1** ([5, 17]) A star-simple drawing that has no crossing-free Hamiltonian cycle.

Our contribution.

Extending the above line of research, we focus on sub-classes of simple drawings and show Conjecture 1.1 to be true for strongly c -monotone drawings (Corollary 3.14) as well as cylindrical drawings (Corollary 3.16). To this end we introduce the following new conjecture.

► **Conjecture 1.2.** *Every simple drawing \mathcal{D} of K_n for $n \geq 2$ contains, for each pair of vertices v_a and v_b in \mathcal{D} , a crossing-free Hamiltonian path with end-vertices v_a and v_b .*

We show that a positive answer to Conjecture 1.2 implies a positive answer to Conjecture 1.1 (Theorem 3.7), which makes Conjecture 1.2 the stronger conjecture. We then verify Conjecture 1.2 for the classes of strongly c -monotone drawings (Theorem 3.13) and cylindrical drawings (Theorem 3.15). Implicitly, to the best of our knowledge, our results also include the first published proofs of Conjecture 1.1 for x -monotone drawings and strongly cylindrical drawings, and cover several more classes of simple drawings.

To better understand the impact of results on sub-classes of simple drawings, we further investigate relations between different such classes (Figure 7). We show that every cylindrical drawing is weakly isomorphic to a c -monotone drawing (Proposition 4.7) and, especially, that every strongly cylindrical drawing is weakly isomorphic to a strongly c -monotone drawing (Theorem 4.9). Furthermore, we provide examples of drawings that are strongly c -monotone but not cylindrical and vice versa (Figure 12). We give a self-contained proof that x -monotone and x -bounded drawings of K_n are the same up to weak isomorphism (Theorem 4.4) and provide an example showing that this is not true anymore for non-complete graphs (Figure 19). Finally, we show that there exist n -shellable drawings of K_n that are not weakly isomorphic to any x -bounded drawing (Observation 4.5), which we do by extending an example from Balko, Fulek, and Kynčl [13] that intended to prove the same statement.

Outline. The rest of the paper is structured as follows: In Section 2 we formally define several classes of simple drawings, provide background information on them, and put them into relation. Section 3 is dedicated to Conjecture 1.2. We first investigate relations between Conjectures 1.1 and 1.2 in Section 3.1. Then we verify Conjecture 1.2 for the above mentioned classes in Section 3.2. Before concluding the paper with a discussion in Section 5, we study relations between different classes of drawings in Section 4. In particular, Section 4.1 contains the results on x -monotone, x -bounded, and n -shellable drawings, whereas in Section 4.2 we show the relations between the different types of cylindrical and c -monotone drawings.

2 An overview on classes of simple drawings

In this section we give a detailed overview on classes of simple drawings that have been studied in the literature. Moreover, we define the class of strongly c -monotone drawings and introduce a clear distinction between two types of cylindrical drawings.

The definitions of most of the following classes depend on a certain representation in the plane and a fixed unbounded cell. However, throughout the paper we consider drawings up to weak isomorphism and only use these special representations to simplify the arguments. Especially, all our statements on crossing-free sub-drawings as well as inclusion relations for drawings \mathcal{D} of a certain class X also hold for all drawings \mathcal{D}' that are weakly isomorphic to any drawing in X . For the sake of readability we do not spell this out though.

We denote the vertices in a simple drawing by v_1, \dots, v_n . For an edge $e = \{v_a, v_b\}$ we assume by convention that $a < b$. Then, using the total order $<$ on the vertices that is induced by their indices, we relate edges to each other: Given two non-incident edges $e = \{v_a, v_b\}$ and $f = \{v_c, v_d\}$, without loss of generality $a < c$, we call e and f

- *separated*, if $a < b < c < d$,
- *linked*¹, if $a < c < b < d$, and
- *nested*, if $a < c < d < b$.

Straight-line drawings. Probably the most studied class of simple drawings are *straight-line drawings* in the plane where each edge is the line segment between its two end-vertices. For straight-line drawings of the complete graph K_n , no three vertices are allowed to lie on a common line, since otherwise edges would pass through vertices. Hence, straight-line drawings of K_n are equivalent to n points in general position in the plane. Further, we may assume without loss of generality that no two vertices share the same x -coordinate.

The *convex straight-line drawing* on n vertices, which we denote by \mathcal{C}_n , is the straight-line drawing of K_n where all vertices are placed in convex position. We say “the” drawing because it is unique up to weak isomorphism. A combinatorial description of \mathcal{C}_n is as follows: There exists an order of the vertices such that two edges cross if and only if they are linked [31].

X -monotone and x -bounded drawings. Let \mathcal{D} be a simple drawing in the plane such that no two vertices have the same x -coordinate. If every vertical line in the plane crosses each edge of \mathcal{D} at most once, then we call \mathcal{D} an *x -monotone drawing*; see Figure 2(a) for an example. If each edge is just contained between the vertical lines through its left and right end-vertex, respectively, then we call \mathcal{D} an *x -bounded drawing*. X -monotone drawings are a natural generalization of straight-line drawings and clearly x -bounded drawings are a generalization of x -monotone drawings. For both x -monotone and x -bounded drawings we can assume, up to strong isomorphism, that all vertices lie on a common horizontal line and we label them v_1, \dots, v_n from left to right. Also, it can be decided in $\mathcal{O}(n^5)$ time whether a given simple drawing of K_n is weakly isomorphic to an x -monotone drawing [8].

2-page-book drawings. A further class of simple drawings are *2-page-book drawings*. They are defined such that all vertices lie on a common horizontal line, the *spine* of the book, and the relative interior of each edge lies completely above or completely below the spine, which defines the two *pages* of the book; see Figure 2(b) for an example. These drawings are

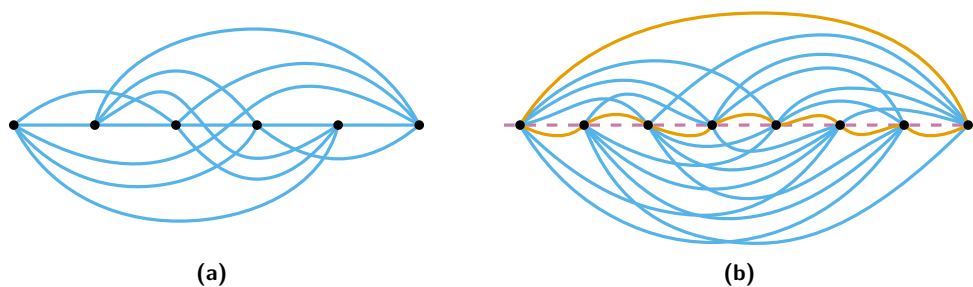
¹ The setting we call *linked* is also called *crossing* in the literature; see Barát, Gyárfás, and Tóth [14]. To avoid confusion with two edges actually having a crossing, we chose a different name though.

closely related to book embeddings, where more than two pages are allowed; see Bernhart and Kainen [16] for details. The following observation can be deduced from the facts that the spine is not crossed by any edge and that two edges in a simple drawing cross at most once.

► **Observation 2.1.** *Two edges in a 2-page-book drawing cross if and only if they lie in the same page and are linked with respect to the vertex order along the spine.*

From this it can easily be seen that a simple drawing of K_n with $n \geq 3$ is weakly isomorphic to a 2-page-book drawing if and only if it contains a completely uncrossed² Hamiltonian cycle. It also follows that the sub-drawing induced by a single page is weakly isomorphic to a sub-drawing of C_n and that each 2-page-book drawing is weakly isomorphic to an x -monotone drawing, for example, by drawing all edges as half-circles in the respective page. Not all 2-page-book drawings are weakly isomorphic to a straight-line drawing though. In particular, there are 2-page-book drawings of K_n , like the one in Figure 2(b), with fewer crossings than any straight-line drawing of K_n can have [3].

For the spine of a 2-page-book drawing we may actually use any simple curve that partitions the plane into two connected components. In particular a circle is possible. In that case we get a circular order π° on the vertices along the spine. However, we can create a linear order π out of π° by choosing some vertex as v_1 and some direction along the circle as increasing. If two edges e and f are linked for one possible choice of π , then e and f are linked for all possible choices of π . Hence, Observation 2.1 holds for any linear order along a circular spine. In contrast to that, every edge pair that is separated for one such linear order π is nested for some other linear order π' and vice versa.



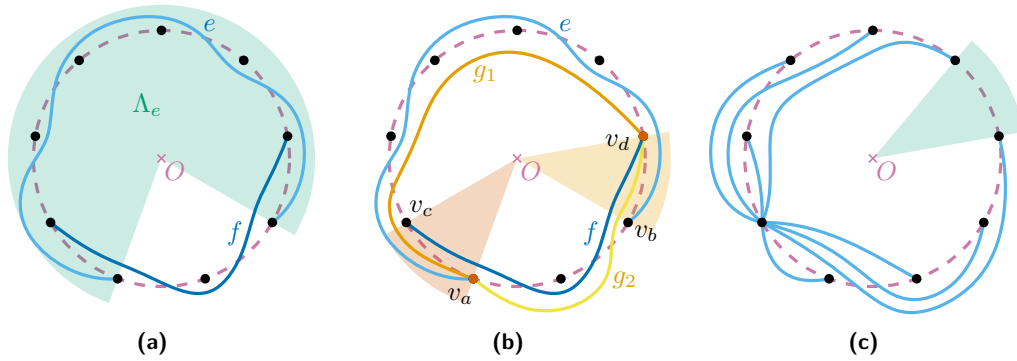
■ **Figure 2** (a) An x -monotone drawing of K_6 . (b) A crossing minimal 2-page-book drawing of K_8 . The spine is drawn dashed purple, the completely uncrossed Hamiltonian cycle is drawn orange.

Shellable drawings. Ábrego et al. [4] introduced the notion of *shellable drawings*. A simple drawing \mathcal{D} is s -shellable if there exists an ordered subset v_1, \dots, v_s of its vertices such that the following two properties hold. First, v_1 and v_s are incident to a common cell F of \mathcal{D} . Second, for each $1 \leq i < j \leq s$, both v_i and v_j are incident to the cell F_{ij} in \mathcal{D}_{ij} , where \mathcal{D}_{ij} is the sub-drawing of \mathcal{D} induced by the vertices v_i, \dots, v_j and F_{ij} is the cell containing F . It is easy to see that every x -bounded drawing of a graph on n vertices is n -shellable (short *shellable*) with the vertices ordered in horizontal direction from left to right and F being the unbounded cell. There exist several generalizations of shellable drawings, for example, *bishellable drawings* (see Ábrego et al. [2]) and *seq-shellable drawings* (see Mutzel and Oettershagen [30]), for which we refer the interested reader to the respective paper.

² We use the term *completely uncrossed* for sub-drawings \mathcal{D}' of a simple drawing \mathcal{D} if no edge of \mathcal{D}' is crossed by any edge of \mathcal{D} . With that we aim to clearly distinguish them from *crossing-free* sub-drawings where the edges of \mathcal{D}' are just not crossed by other edges of \mathcal{D}' .

C-monotone drawings. To characterize the next class, we follow the definition given in [6]. An edge e in a simple drawing is *c-monotone with respect to a point O of the plane* if every ray starting at O crosses e at most once. We call a simple drawing \mathcal{D} in the plane a *c-monotone drawing* if all edges in \mathcal{D} are c-monotone with respect to a common point O and no two vertices lie on a common ray starting at O . The letter “c” in c-monotone is meant as an abbreviation of “circularly” and “ O ” stands for “the origin”. When we consider a c-monotone drawing \mathcal{D} , we assume that a suitable fixed point O is given with it. We can also assume, up to strong isomorphism, that all vertices of \mathcal{D} lie on a common circle with center O .

Apart from being a generalization of x -monotone drawings, c-monotone drawings are of interest for the following reason: In a simple drawing \mathcal{D} of K_n the star \mathcal{S}_v of any vertex v can be extended by a plane spanning tree \mathcal{T} on the $n - 1$ vertices of \mathcal{D} excluding v such that also $\mathcal{S}_v + \mathcal{T}$ is plane; see García, Pilz, and Tejel [20, Corollary 3.4]. The structure $\mathcal{S}_v + \mathcal{T}$ then either contains large plane sub-drawings itself (like matchings, paths, and cycles) or enforces the existence of a large c-monotone sub-drawing in \mathcal{D} . See Fulek [18, Proof of Theorem 1.1] for more details on the idea behind this.



■ **Figure 3** (a) The wedge Λ_e (shaded seagreen) of the lightblue edge e . The darkblue edge f together with e covers the plane, which is not allowed in a strongly c-monotone drawing. (b) Illustration of Lemma 2.2: A pair of non-incident edges (blue) covering the plane enforces a pair of incident edges covering the plane; either with the darker (g_1) or with the lighter (g_2) version of g . (c) Condition 3 of Lemma 2.2: The star is not crossed by any ray within the seagreen wedge.

Strongly c-monotone drawings. Given an edge e in a c-monotone drawing, we define the *wedge Λ_e of e* to be the closed wedge with apex O that has the rays from O through the end-vertices of e as its boundary and contains e . For a set of edges E we say that the edges in E *cover the plane* if the union of all their wedges is the whole plane. See Figure 3(a) for an illustration of those terms. Clearly it needs at least two edges to cover the plane. This leads us to the definition of strongly c-monotone drawings. If no pair of edges in a c-monotone drawing \mathcal{D} covers the plane, then we call \mathcal{D} *strongly c-monotone*. Equivalently, a c-monotone drawing is strongly c-monotone if, for each pair of edges e and f in \mathcal{D} , there exists a ray starting at O that crosses neither e nor f . Note that all x -monotone drawings are also strongly c-monotone: Choose O sufficiently far below the drawing such that every ray starting at O crosses each edge of \mathcal{D} at most once; then every ray pointing away from \mathcal{D} does not cross any edge of \mathcal{D} . For K_n , strong c-monotonicity can be characterized as follows.

► **Lemma 2.2.** *Let \mathcal{D} be a c-monotone drawing of K_n . Then the following are equivalent:*

1. *No pair of edges e and f in \mathcal{D} covers the plane.*
2. *No pair of incident edges e and f in \mathcal{D} covers the plane.*
3. *No star in \mathcal{D} covers the plane.*

Proof. Clearly, the first condition implies the second. For the second implying the first, assume that two non-incident edges $e = \{v_a, v_b\}$ and $f = \{v_c, v_d\}$ cover the plane. Then the wedges Λ_e of e and Λ_f of f intersect in two wedges with apex O . Without loss of generality, they do so between v_a and v_c , and between v_b and v_d ; see Figure 3(b) for an illustration. We now consider the edge $g = \{v_a, v_d\}$. Either g is contained in Λ_e , then the incident edges g and f cover the plane. Or g is contained in Λ_f and the incident edges g and e cover the plane. This establishes equivalence of the first two conditions. The equivalence of conditions two and three is straightforward: A star covers the plane if and only if two of its edges, which are incident edges, cover the plane. ◀

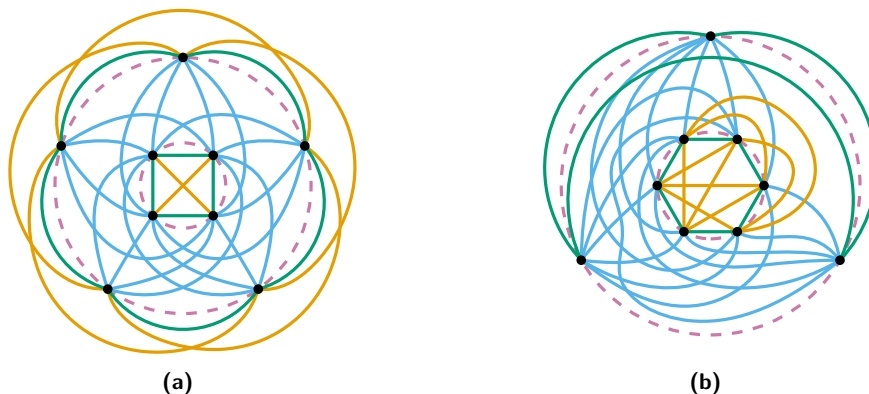
Especially the third condition is quite easy to verify, at least by hand, only requiring n checks instead of the $\mathcal{O}(n^4)$ checks when considering all edge pairs; see also Figure 3(c).

Cylindrical drawings. Following the definition introduced in [4], we call a simple drawing *cylindrical* if all vertices lie on two concentric circles and no edge crosses any of these two circles. The name “cylindrical” comes from picturing it as being drawn on the surface of a cylinder: The vertices are placed on the two rims and each edge lies completely in the top lid, the lateral face, or the bottom lid. This is a generalization of Hill’s conjectured crossing minimal drawing of K_n , which we denote by \mathcal{Z}_n . It can be achieved by placing half of the vertices equally spaced on each rim and drawing the edges as geodesics. See Figure 4(a) for a depiction and Harary and Hill [24] or Guy, Jenkyns, and Schaer [23] for more details.

For cylindrical drawings in the plane, as defined above, we call the area outside the outer circle, between the two circles, and inside the inner circle *outer*, *lateral*, and *inner face*, respectively. Additionally, we call edges connecting two vertices from different circles *lateral edges* and edges connecting two vertices on the inner or outer circle *inner* or *outer*, respectively, *circle edges*. In particular, we call circle edges connecting two neighboring vertices on their circle *rim edges*. See Figure 4 for some illustrations of these terms.

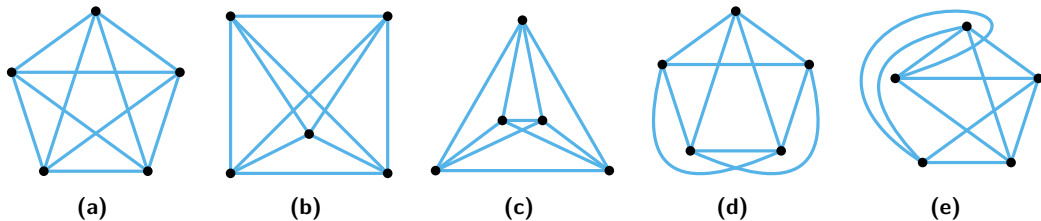
Because the two concentric circles are not allowed to be crossed by any edge, all lateral edges have to lie in the lateral face. In contrast to that, however, inner (outer, respectively) circle edges can lie either in the inner (outer, respectively) face or in the lateral face. Also, the following observation is implied by the definition.

► **Observation 2.3.** *Let \mathcal{D} be a cylindrical drawing. Then the sub-drawing of \mathcal{D} induced by all vertices on one of the circles is strongly isomorphic to a 2-page-book drawing.*



■ **Figure 4** (a) Hill’s drawing \mathcal{Z}_9 drawn in the plane. The two concentric circles are drawn violet, the rim edges are seagreen, all other circle edges are orange, and the lateral edges are lightblue. (b) An arbitrary (not strongly) cylindrical drawing of K_9 with the same color coding.

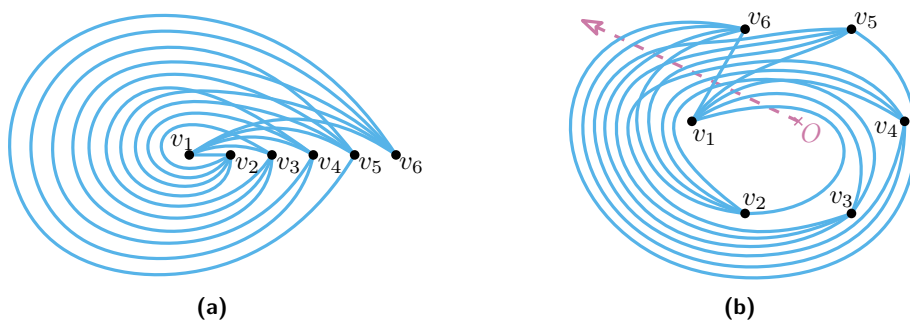
Strongly cylindrical drawings. If in a cylindrical drawing all inner circle edges lie in the inner face and all outer circle edges lie in the outer face, then we call the drawing *strongly cylindrical*. Especially, Hill’s drawing \mathcal{Z}_n is strongly cylindrical. In the literature the term “cylindrical drawing” is also used for what we call strongly cylindrical drawing. Further, Ruiz-Vargas [33] defines arbitrary drawings on the surface of an infinite open cylinder as “cylindrical drawings” and his “monotone cylindrical drawings” are equivalent to our c-monotone drawings.



■ **Figure 5** The five non-isomorphic simple drawings of K_5 : (a) The convex straight-line drawing \mathcal{C}_5 . (a) – (c) The three types of straight-line drawings of K_5 . (c) In some sense Hill’s drawing \mathcal{Z}_5 . (e) The twisted drawing \mathcal{T}_5 . (a) and (e) The two crossing maximal drawings of K_5 .

Crossing maximal drawings. In a simple drawing \mathcal{D} of K_n any sub-drawing induced by 4 vertices can admit at most one crossing. If every 4-tuple of vertices induces a crossing and hence \mathcal{D} contains exactly $\binom{n}{4}$ crossings, then we call \mathcal{D} *crossing maximal*. For example, the drawings of K_5 in Figures 5(a) and 5(e), and in general \mathcal{C}_n , are crossing maximal.

Twisted and generalized twisted drawings. In their study of crossing maximal drawings, Harborth and Mengersen [27] introduced another special drawing of K_n , which generalizes the drawing in Figure 5(e). Harborth [25] further analyzed that drawing in the context of empty triangles and Pach, Solymosi, and Tóth [31] later named it twisted. A simple drawing of K_n is a *twisted drawing* if there exists an order on its vertices such that two edges cross if and only if they are nested. This uniquely defines the twisted drawing of K_n , which we denote by \mathcal{T}_n , up to weak isomorphism and we use this defining order to label its vertices from v_1 to v_n . One possible way of realizing this special crossing property is shown in Figure 6(a). Also, the drawing in Figure 2(a) is actually an x -monotone representation of \mathcal{T}_6 ; for $n \geq 7$ this is not possible anymore. However, \mathcal{T}_n can be drawn c-monotone such that there exists a ray starting at O that crosses all edges; Figure 6(b) shows an example. This is the defining property for the following generalization of twisted drawings [6]: A c-monotone drawing \mathcal{D} of K_n is called *generalized twisted* (short g -twisted) if there exists a ray starting at O that crosses all edges of \mathcal{D} . It has been shown [7] that a simple drawing \mathcal{D} of K_n , for $n \geq 7$, is



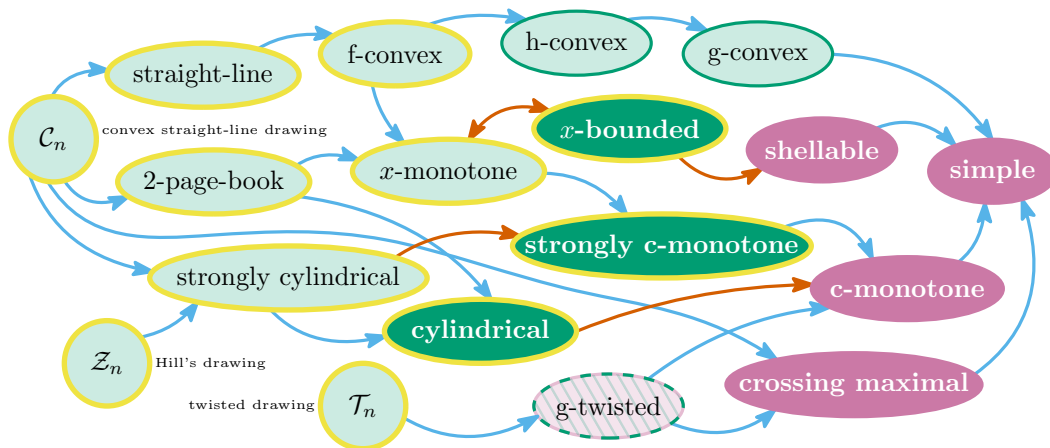
■ **Figure 6** The twisted drawing \mathcal{T}_6 drawn (a) in the usual way and (b) as a c-monotone drawing.

weakly isomorphic to a g -twisted drawing if and only if every sub-drawing of \mathcal{D} induced by 5 vertices is weakly isomorphic to the twisted drawing \mathcal{T}_5 . This especially implies that all g -twisted drawings are crossing maximal drawings. For more information on g -twisted drawings we refer the interested reader to [6, 7, 21].

Generalized convex drawings. Arroyo, McQuillan, Richter, and Salazar [11] introduced the following class of simple drawings: Every triangle in a simple drawing separates the plane into two connected components, the closures of which are called the *sides* of the triangle. A side S of a triangle is called *convex* if, for each pair of vertices v_a and v_b in S , the edge $\{v_a, v_b\}$ is completely contained in S . We call a simple drawing \mathcal{D} of K_n *generalized convex*³ (short *g-convex*) if every triangle in \mathcal{D} has a convex side. It has been shown [11, Theorem 2.6] that a drawing is g -convex if and only if every sub-drawing induced by 5 vertices is weakly isomorphic to one of the three types of straight-line drawings of K_5 ; see Figures 5(a)–5(c).

Further, there exists the sub-class of *hereditarily convex drawings* (short *h-convex*) where the convex side of triangles is inherited through inclusion. Arroyo, Richter, and Sunohara [12] showed that, for K_n , this is equivalent to so-called *pseudospherical drawings*, a generalization of drawings on the sphere with the edges drawn as geodesics. Going down one more step, an h -convex drawing \mathcal{D} is called *face convex* (short *f-convex*) if there exists a cell F in \mathcal{D} such that for every triangle in \mathcal{D} the side not containing F is convex. For K_n , Arroyo, McQuillan, Richter, and Salazar [10] proved this to be equivalent to the well-known *pseudolinear drawings*. And those are in turn a generalization of straight-line drawings and a sub-class of x -monotone drawings. For more information on g -convex drawings and their sub-classes see [10, 11, 12].

Overview. We finish this section with an overview on all the introduced drawing classes and an outlook on what we will show in the following sections. Since a picture is worth a thousand words, we do so with Figure 7.



■ **Figure 7** An overview on different classes of simple drawings of K_n . Single arrows indicate that the source class is a proper subset of the target class, regarding weak isomorphism; a double arrow means equality. Each darkorange arrow indicates a result shown in this work. The seagreen disks mark classes for which Conjecture 1.1 is (now) proven; we focus on the darker seagreen ones. All classes for which we verify Conjecture 1.2 are emphasized with a yellow boundary. For violet classes both conjectures are still open, while for g -twisted drawings Conjecture 1.1 is partially proven.

³ Arroyo et al. [10, 11, 12] called them just “convex drawings”. But because in these drawings both sides of a triangle can be convex we prefer to call them “generalized convex drawings”.

3 Crossing-free Hamiltonian paths between all vertex pairs

This section is devoted to showing that Conjecture 1.2 is indeed a strengthening of Conjecture 1.1 and to proving these conjectures for strongly c -monotone drawings and cylindrical drawings. Since the question regarding a crossing-free Hamiltonian path between each pair of vertices is new (Conjecture 1.2), it is prudent to check whether this actually makes sense. It can be shown that in every straight-line drawing of K_n all those crossing-free Hamiltonian paths actually exist. The main idea for this is to visit all vertices in circular order around some start-vertex. We leave a formal proof to the reader because the statement also follows from more general results later in this paper.

As another sanity check, using our rotation system database [1], we can confirm Conjecture 1.2 for all simple drawings on $n \leq 9$ vertices. Moreover, Manfred Scheucher (personal communication) used the SAT framework from [15] to confirm Conjecture 1.2 for all simple drawings on $n \leq 10$ vertices and for all g -convex drawings on $n \leq 12$ vertices.

A problem related to prespecifying both end-vertices of a crossing-free Hamiltonian path \mathcal{P} would be to fix a certain edge to be part of \mathcal{P} . In straight-line drawings of K_n this is easily possible because the supporting line of an edge splits the vertex set into two parts with disjoint convex hulls. Moreover, in Theorem 4.1 of the arXiv version of [15], Bergold, Felsner, M. Reddy, and Scheucher showed that even in g -convex drawings \mathcal{D} of K_n there exists, for each edge e in \mathcal{D} , a crossing-free Hamiltonian path containing e . However, in an arbitrary simple drawing of K_n this is not possible. For example, choose the edge $\{v_1, v_n\}$ in the twisted drawing \mathcal{T}_n : Since it crosses all non-incident edges, it cannot be part of any crossing-free path with more than three edges.

3.1 All paths conjecture implies cycle conjecture

As a first main result of this paper we establish a connection between Conjectures 1.1 and 1.2. We start by observing that every completely uncrossed edge in a simple drawing \mathcal{D} can be added to a crossing-free sub-drawing of \mathcal{D} without introducing any new crossings.

► **Observation 3.1.** *Let \mathcal{D} be a simple drawing of K_n with $n \geq 3$ that contains a completely uncrossed edge $e = \{v_a, v_b\}$ and a crossing-free Hamiltonian path with end-vertices v_a and v_b . Then \mathcal{D} contains a crossing-free Hamiltonian cycle.*

This observation establishes that Conjecture 1.2 is at least as strong as Conjecture 1.1 for all drawings containing a completely uncrossed edge. This includes, for example, x -bounded drawings because with the vertices ordered from left to right, two separated edges can never cross in an x -bounded drawing. Especially, we have the following.

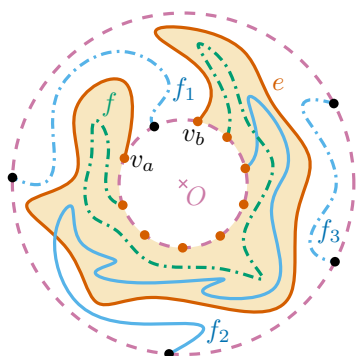
► **Observation 3.2.** *Let \mathcal{D} be an x -bounded drawing of K_n with vertices v_1, \dots, v_n labeled from left to right. Then the edges $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$ are completely uncrossed.*

We next show that also every cylindrical drawing contains completely uncrossed edges. For a circle edge $e = \{v_a, v_b\}$ in the lateral face and end-vertices on circle C , we consider the unique area F that is completely contained in the lateral face, and only bounded by e and a part of C ; see Figure 8 for an example illustration. We say that e guards a vertex v on C if v lies on the boundary of F , that is, e guards all vertices between and including v_a and v_b on C , either in clockwise or in counter-clockwise direction. In particular, e always guards its own end-vertices. Similar to Observation 2.1 we get the following.

► **Observation 3.3.** *Let e be a circle edge in the lateral face of a cylindrical drawing \mathcal{D} . Let f be another edge of \mathcal{D} that is non-incident to e and also lies in the lateral face of \mathcal{D} . Then e and f cross if and only if e guards exactly one end-vertex of f .*

We further say that e guards another edge f if f also lies in the lateral face and both its end-vertices are guarded by e , which is equivalent to f lying completely in F . It therefore follows that two edges cannot mutually guard one another.

► **Observation 3.4.** *Let e be a circle edge in the lateral face of a cylindrical drawing \mathcal{D} , let V_e be the set of vertices guarded by e , and let e guard another circle edge f . Then the set of vertices guarded by f is a proper subset of V_e .*



■ **Figure 8** The orange circle edge e guards all orange vertices and the seagreen edge f . The area F is shaded lightorange. In support of Observation 3.3, three more examples for edges f_i are drawn in lightblue. The solid edge f_2 crosses e , whereas all dash dotted edges do not cross e .

With these observations we can now show the following fundamental structural statement for cylindrical drawings, which we also use later for most of the results on these drawings.

► **Lemma 3.5.** *Let \mathcal{D} be a cylindrical drawing. Then all but at most two rim edges in \mathcal{D} , at most one per circle, are completely uncrossed.*

Proof. Consider, without loss of generality, the inner circle. First, if a rim edge e lies in the inner face, then e is uncrossed by Observations 2.1 and 2.3. Further, if e lies in the lateral face, then there are two more cases. Either e only guards its own end-vertices, then it is uncrossed by Observation 3.3. Or e guards all vertices on the inner circle. Then e can have crossings but e also guards all other inner rim edges that lie in the lateral face. Especially, by Observation 3.4, all those rim edges guarded by e only guard their own end-vertices. Hence, by the first two cases, all other inner rim edges are uncrossed. ◀

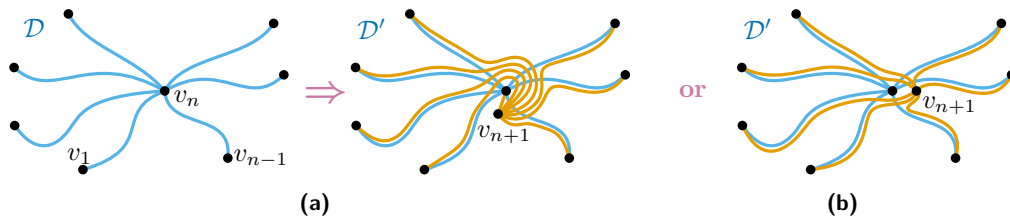
► **Corollary 3.6.** *For every cylindrical or x -bounded drawing, as well as every other drawing containing a completely uncrossed edge, Conjecture 1.2 implies Conjecture 1.1.*

Unfortunately, not all simple drawings of K_n have completely uncrossed edges. Harborth and Mengersen [26] already observed that for every $n \geq 8$ there exist simple drawings of K_n where every edge is crossed. Moreover, Kynčl and Valtr [29] showed that there are simple drawings of K_n where every edge is crossed by as many as $\Omega(n^{3/2})$ other edges. However, we can still prove a relation between Conjectures 1.1 and 1.2 for those drawings, by considering all simple drawings of K_n at once.

► **Theorem 3.7.** *Let $n \geq 3$ be fixed. If Conjecture 1.2 is true for all simple drawings of K_{n+1} , then Conjecture 1.1 is true for all simple drawings of K_n .*

Proof. Let \mathcal{D} be an arbitrary simple drawing of K_n for some fixed $n \geq 3$ and assume Conjecture 1.2 is true for all simple drawings of K_{n+1} . Consider the vertex $v_n \in \mathcal{D}$ and, especially, the star of v_n . Let, without loss of generality, the rotation of v_n be v_1, \dots, v_{n-1} in clockwise order. We produce a simple drawing \mathcal{D}' of K_{n+1} by duplicating v_n in the following way (see Figure 9(a) for an illustration of the construction). We place a new vertex v_{n+1} close to v_n into the cell of \mathcal{D} that is incident to v_n and has the edges $\{v_n, v_{n-1}\}$ and $\{v_n, v_1\}$ on its boundary. We then add, for $1 \leq i \leq n - 1$, the edges $\{v_{n+1}, v_i\}$ by starting at v_{n+1} , going around v_n in counter-clockwise direction until we reach the edge $\{v_n, v_i\}$, and then following $\{v_n, v_i\}$ until we reach v_i . That way, the edge $\{v_{n+1}, v_i\}$ crosses exactly all edges $\{v_n, v_j\}$, for $i < j < n$, and all edges that are crossed by $\{v_n, v_i\}$ in \mathcal{D} . Finally, we connect v_n and v_{n+1} by a completely uncrossed edge.

By assumption, \mathcal{D}' contains a crossing-free Hamiltonian path \mathcal{P} with end-vertices v_n and v_{n+1} . Because $n + 1 \geq 4$, the path \mathcal{P} uses neither both of $\{v_i, v_n\}$ and $\{v_i, v_{n+1}\}$ for any vertex v_i in \mathcal{D}' with $1 \leq i \leq n - 1$ nor the edge $\{v_n, v_{n+1}\}$. Therefore, identifying v_n and v_{n+1} in \mathcal{P} produces a Hamiltonian cycle \mathcal{C} in \mathcal{D} . Since all edges $\{v_i, v_{n+1}\}$ have the same crossings as $\{v_i, v_n\}$, except for crossings with edges incident to v_n and v_{n+1} , the cycle \mathcal{C} is also crossing-free. And as \mathcal{D} was arbitrary, this finishes the proof. ◀



■ **Figure 9** (a) The basic construction to add a vertex to a simple drawings of K_n . (b) A possible different placement of v_{n+1} together with a different routing of its incident edges.

We remark that the described construction was also used by Harborth and Mengersen [27] to inductively produce crossing maximal drawings of K_n . Further, if we want an analogous statement of Theorem 3.7 for only a sub-class of simple drawings, then we have to make sure that the drawing \mathcal{D}' can be constructed in such a way that it also lies in that sub-class. For that, we actually have several possibilities to vary the construction without changing the essential crossing properties of it. First, the vertex v_{n+1} can be placed in any of the $n - 1$ cells incident to v_n . Second, there are n different ways of how to split the edges incident to v_{n+1} into two sets, regarding whether they go around v_n in clockwise or counter-clockwise direction. See Figure 9(b) for an example. In particular, for strongly c-monotone drawings, we can place v_{n+1} directly next to v_n in circular order around O and draw all edges such that they are c-monotone with respect to O . It then follows that also \mathcal{D}' is strongly c-monotone (by Lemma 2.2, we only need to check each star in \mathcal{D}').

► **Corollary 3.8.** *If Conjecture 1.2 is true for all strongly c-monotone drawings, then also Conjecture 1.1 is true for all strongly c-monotone drawings.*

We note that, for example, for the classes of g-twisted or g-convex drawings, it is not immediately clear whether an analogous statement holds.

3.2 Proving the all paths conjecture for sub-classes of simple drawings

In this section we prove Conjecture 1.2 for strongly c -monotone drawings and for cylindrical drawings. As a basis for both cases, we first verify the conjecture for x -monotone drawings. We formulate the following definitions for x -bounded drawings because we will need them later in this more general setting. Given an edge $e = \{v_a, v_b\}$ in an x -bounded drawing \mathcal{D} , we consider the area F between and including the vertical lines through v_a and v_b , respectively. The edge e splits F into two connected components, the closures thereof we call the two *sides* of e . If a sub-drawing of \mathcal{D} , in particular an edge or a vertex, is completely contained in one side of e , we say that it lies *above* or *below* e . The following observation can be deduced from the definition of x -bounded drawings and the fact that two edges in a simple drawing have at most one point in common.

► **Observation 3.9.** *Let e and f be two edges of an x -bounded drawing \mathcal{D} such that both end-vertices of f lie in the same side of e . Then the whole edge f lies in that side.*

With this, we can inductively combine crossing-free paths on certain sub-drawings of \mathcal{D} to a crossing-free Hamiltonian path in \mathcal{D} . For convenience we state the following proposition only for x -monotone drawings. An analogous statement for x -bounded drawings will later follow from Theorem 4.4.

► **Proposition 3.10.** *Let \mathcal{D} be an x -monotone drawing of K_n and let v_a and v_b be two vertices in \mathcal{D} . Then \mathcal{D} contains a crossing-free Hamiltonian path with end-vertices v_a and v_b .*

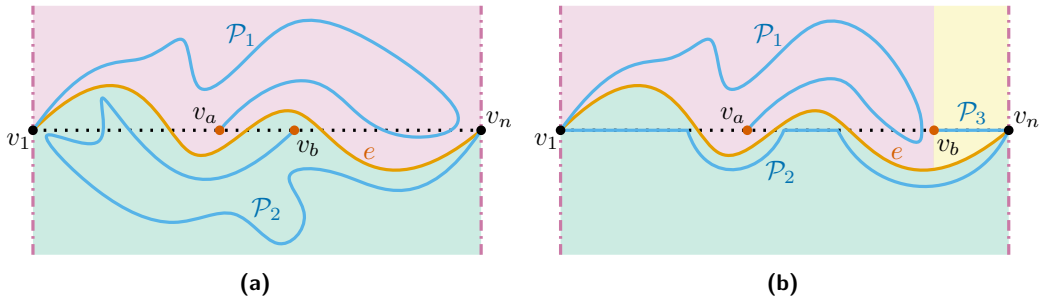
Proof. We prove the statement by induction on n . For $n \leq 3$, it is obviously true. For $n > 3$, let the vertices v_1, \dots, v_n , as usual, be labeled from left to right in horizontal direction.

Let first v_a and v_b , with $a, b \notin \{1, n\}$, lie in different sides of the edge $e = \{v_1, v_n\}$, without loss of generality, v_a above and v_b below. See Figure 10(a) for an illustration of the following. The sub-drawing induced by all vertices above e excluding v_n and the sub-drawing induced by all vertices below e excluding v_1 are proper sub-drawings of \mathcal{D} and clearly x -monotone. Hence, by the induction hypothesis, there exists a crossing-free path \mathcal{P}_1 with end-vertices v_a and v_1 , visiting all vertices above e except v_n , and another crossing-free path \mathcal{P}_2 with end-vertices v_n and v_b , visiting all vertices below e except v_1 . Further, by Observation 3.9, e separates \mathcal{P}_1 and \mathcal{P}_2 . Therefore, combining \mathcal{P}_1 and \mathcal{P}_2 via e creates a crossing-free Hamiltonian path with end-vertices v_a and v_b .

Let next v_a and v_b , with $1 < a < b$, lie in the same side of e , without loss of generality, above e . See Figure 10(b) for an illustration. Then, by the induction hypothesis and similar to before, there exists a crossing-free path \mathcal{P}_1 with end-vertices v_a and v_1 , visiting all vertices above e that lie to the left of v_b , another crossing-free path \mathcal{P}_2 with end-vertices v_1 and v_n , visiting all vertices below e , and a third crossing-free path \mathcal{P}_3 with end-vertices v_n and v_b , visiting all vertices between v_n and v_b that lie above e . The path \mathcal{P}_3 may just be a single vertex, which happens if $b = n$, and at most \mathcal{P}_2 can contain e , which happens if there are no vertices strictly below e . Further, by x -monotonicity, \mathcal{P}_1 cannot cross \mathcal{P}_3 and, by Observation 3.9, \mathcal{P}_2 cannot cross any of the other two paths. Therefore, combining all three paths creates a crossing-free Hamiltonian path with end-vertices v_a and v_b . The case $a < b < n$ works similarly with the arguments being right-left reversed.

Finally, if $v_a = v_1$ and $v_b = v_n$, then the path in given order from left to right is a crossing-free Hamiltonian path with end-vertices v_a and v_b . ◀

We next show that strongly c -monotone drawings behave locally very much like x -monotone drawings. Recall the definition of the wedge Λ_e of an edge e from Section 2 and be aware that the following lemma does in general not hold for (non-strongly) c -monotone drawings.



■ **Figure 10** The two cases for finding a crossing-free Hamiltonian path between two given vertices v_a and v_b in an x -monotone drawing of K_n when v_a and v_b lie (a) in different sides or (b) in the same side of the edge $e = \{v_1, v_n\}$. Edges are drawn orange while longer paths are lightblue.

► **Lemma 3.11.** *Let e be an edge in a strongly c -monotone drawing \mathcal{D} . Then the sub-drawing \mathcal{D}_e of \mathcal{D} , induced by all vertices in the wedge Λ_e of e , is fully contained in Λ_e and strongly isomorphic to an x -monotone drawing.*

Proof. Suppose that an edge f with both end-vertices in Λ_e leaves Λ_e through one of its boundary rays. Then f has to encircle O and enter Λ_e again through its other boundary ray. Consequently e and f cover the plane, a contradiction to strong c -monotonicity. Hence, \mathcal{D}_e is fully contained in Λ_e . Furthermore, there exists a homeomorphism of the plane that maps \mathcal{D}_e to an x -monotone drawing, for example, by first rotating the plane accordingly around O and then moving O to infinity through the complement of Λ_e . ◀

For a c -monotone drawing \mathcal{D} with origin O and two consecutive vertices in the circular order around O , we denote the closed wedge with apex O that has the rays from O through these two vertices as its boundary and does not contain any other vertex of \mathcal{D} as a *gap* of \mathcal{D} . Further, we call an edge between two consecutive vertices a *gap edge* and associate it with the respective gap. Then the wedge Λ_e of a gap edge e is either identical to its gap or to the closure of the complement of its gap. Together with Lemma 3.11 this implies the following central property of strongly c -monotone drawings.

► **Corollary 3.12.** *Let \mathcal{D} be a strongly c -monotone drawing. Then either all gap edges are contained in their respective gaps or \mathcal{D} is strongly isomorphic to an x -monotone drawing.*

Proof. Assume there exists a gap edge e that is not contained in its gap. Then the wedge Λ_e of e is the closure of the complement of its gap and hence contains all vertices of \mathcal{D} . Consequently, by Lemma 3.11, \mathcal{D} is strongly isomorphic to an x -monotone drawing. ◀

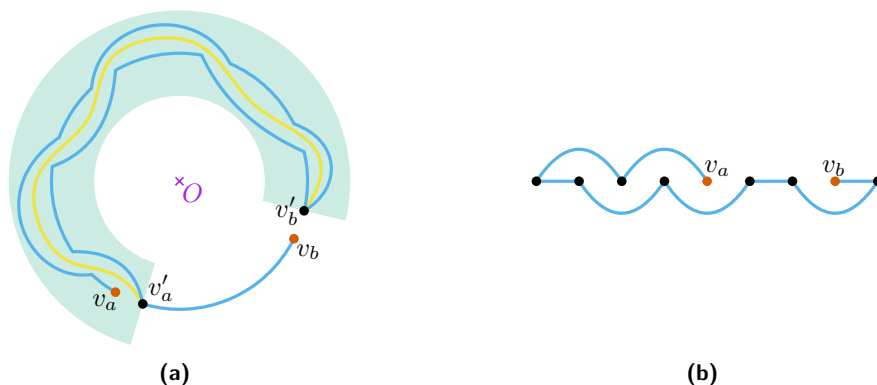
Corollary 3.12 implies that for every strongly c -monotone drawing that is not strongly isomorphic to an x -monotone drawing, its gap edges form a crossing-free Hamiltonian cycle.

► **Theorem 3.13.** *Let \mathcal{D} be a strongly c -monotone drawing of K_n and let v_a and v_b be two vertices in \mathcal{D} . Then \mathcal{D} contains a crossing-free Hamiltonian path with end-vertices v_a and v_b .*

Proof. If \mathcal{D} is strongly isomorphic to an x -monotone drawing, then the statement follows from Proposition 3.10. So we can assume, by Corollary 3.12, that all gap edges are contained in their respective gaps.

We distinguish two cases. If v_a and v_b are neighbors in the circular order around O , then the Hamiltonian path between them that consists only of gap edges is crossing-free. Otherwise, let v'_a and v'_b be the vertices in clockwise circular order around O directly before

v_a and v_b , respectively; see Figure 11(a) for an illustration. Let further, without loss of generality, the wedge Λ_e of the edge $e = \{v'_a, v'_b\}$ contain v_a . Then, by Lemma 3.11, the sub-drawing \mathcal{D}_e induced by the vertices in Λ_e is strongly isomorphic to an x -monotone drawing. Hence, by Proposition 3.10, there exists a crossing-free Hamiltonian path in \mathcal{D}_e with end-vertices v_a and v'_a . Finally, extending that path from v'_a to v_b by gap edges in counter-clockwise order around O creates a Hamiltonian path with end-vertices v_a and v_b in \mathcal{D} which, by Lemma 3.11 and the above assumption, is crossing-free. ◀



■ **Figure 11** (a) Construction of a crossing-free Hamiltonian path between two given vertices in a strongly c -monotone drawing of K_n . (b) A crossing-free Hamiltonian path between two given vertices of a twisted drawing. For better readability, the edges are not drawn in the usual way but instead indicated figuratively.

By Corollary 3.8, this also proves Conjecture 1.1 for strongly c -monotone drawings.

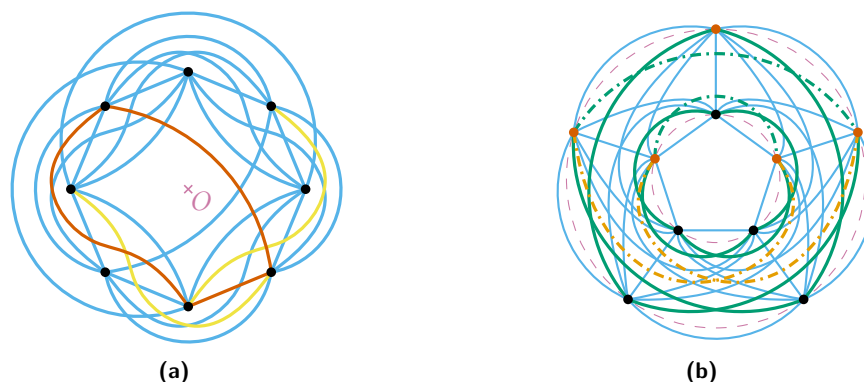
► **Corollary 3.14.** *Let \mathcal{D} be a strongly c -monotone drawing of K_n with $n \geq 3$. Then \mathcal{D} contains a crossing-free Hamiltonian cycle.*

Let us mention at this point that Conjectures 1.1 and 1.2 also hold for the twisted drawing \mathcal{T}_n . For constructing a crossing-free Hamiltonian path or cycle, we just have to make sure not to use any pair of nested edges. Figure 11(b) indicates a possible crossing-free Hamiltonian path between two given vertices in a twisted drawing, using only edges between vertices that are at most at distance two from each other in the defining vertex order.

In Figure 12(a) we give an example of a strongly c -monotone drawing that is not weakly isomorphic to any x -monotone or cylindrical drawing, by Observation 3.2 and Lemma 3.5, because it does not have any completely uncrossed edges. It is also not g -convex since it contains triangles without a convex side. In particular, this means that Corollary 3.14 verifies Conjecture 1.1 for more simple drawings than the ones for which it was known before.

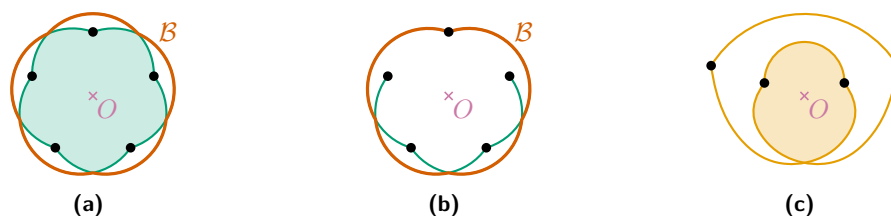
Furthermore, Figure 12(b) shows an example of a cylindrical drawing \mathcal{D}_{10} that is neither g -convex nor weakly isomorphic to any strongly c -monotone drawing. For the not g -convex part, for example, the sub-drawing induced by the five darkorange vertices is weakly isomorphic to the non-straight-line drawing of K_5 in Figure 5(d). To show that \mathcal{D}_{10} is not weakly isomorphic to any strongly c -monotone drawing, we need more involved arguments. We give the basic ideas in the following.

First, imagine a c -monotone drawing being mapped onto the sphere such that the origin O and the point at infinity, which we denote by P_∞ , are opposite poles. Then O and P_∞ behave exactly the same with regard to the definition of c -monotonicity and, thus, can be interchanged. Next, we consider sub-structures of simple drawings that restrict where O and



■ **Figure 12** (a) A strongly c -monotone drawing of K_8 that is neither x -monotone nor cylindrical nor g -convex. The darkorange triangle has no convex side, which is witnessed by the two yellow edges. (b) A cylindrical drawing \mathcal{D}_{10} of K_{10} that is neither strongly c -monotone nor g -convex.

P_∞ must be placed such that a drawing \mathcal{D} can be c -monotone. The first structure, which we call a *cloud*⁴, is a simple closed curve \mathcal{B} consisting of parts of the relative interior of edges in \mathcal{D} such that the end-vertices of all involved edges lie on the same side of \mathcal{B} , which we call the *inside* of the cloud, and no end-vertex lies on \mathcal{B} . Thereby the sides of \mathcal{B} are defined analogously to the sides of a triangle. Figure 13(a) shows a cloud that also appears in \mathcal{D}_{10} . The second structure, which we call a *fish*, is defined the same way as a cloud but with one vertex of \mathcal{D} lying on the simple closed curve \mathcal{B} . The edges forming the cloud in Figure 13(a) in fact also form five fishes; one of them is shown in Figure 13(b). The intersection of the insides of these five fishes, which we call the *center cell* of the cloud, is shaded seagreen in Figure 13(a). For a fish or cloud \mathcal{B} in a c -monotone drawing, either O or P_∞ must lie in its inside because \mathcal{B} contains at most one vertex, where it could change its circular direction around O without forcing an edge to violate c -monotonicity. If both O and P_∞ would lie in the outside, then \mathcal{B} would have to change its direction at least twice. Now, observe that \mathcal{D}_{10} actually contains two such clouds, which are formed by the seagreen edges in Figure 12(b) and have disjoint insides. Hence, O must lie in the center cell of one of the two clouds and P_∞ must lie in the other. Finally, we consider *pretzels*, which are cycles of length 4 that encircle some area in the plane twice; shaded orange in Figure 13(c). If, in a c -monotone drawing \mathcal{D} , O lies in that area, then some pair of edges of the pretzel will cover the plane and consequently \mathcal{D} is not strongly c -monotone. By the above arguments and a result from Gioan [22, Theorem 3.10] stating that weakly isomorphic drawings of K_n can be transformed into each other by triangle flips, this is the case for every c -monotone drawing that is weakly isomorphic to \mathcal{D}_{10} ; the pretzel in Figure 12(b) is drawn dash dotted.



■ **Figure 13** The three structures we use to show that \mathcal{D}_{10} from Figure 12(b) is not weakly isomorphic to any strongly c -monotone drawing: (a) A cloud, (b) a fish, and (c) a pretzel.

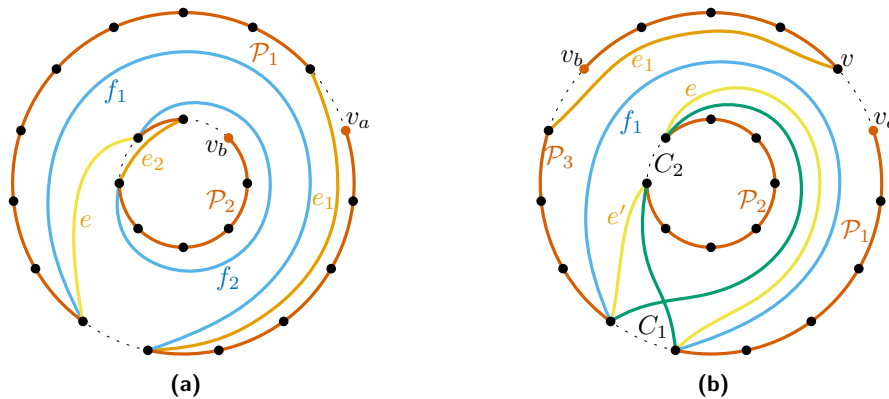
⁴ Arroyo, Bensmail, and Richter [9] used similar structures to characterize pseudolinear drawings.

Since we now know that not all cylindrical drawings are weakly isomorphic to a strongly c-monotone drawing, we conclude this section by verifying Conjecture 1.2 for cylindrical drawings. In our proof, we use a similar idea as in a “classroom proof” of Conjecture 1.1 for *strongly cylindrical drawings*.

► **Theorem 3.15.** *Let \mathcal{D} be a cylindrical drawing of K_n and let v_a and v_b be two vertices in \mathcal{D} . Then \mathcal{D} contains a crossing-free Hamiltonian path with end-vertices v_a and v_b .*

Proof. By Lemma 3.5, all but at most one rim edges per circle are completely uncrossed. Moreover, if a rim edge f has crossings, then f lies in the lateral face and guards all vertices on its circle. We now distinguish two situations.

Let first v_a and v_b lie on different circles of \mathcal{D} . See Figure 14(a) for an illustration of the following. We construct a crossing-free path \mathcal{P}_1 by starting at v_a and visiting all vertices on the same circle in clockwise order, if possible. If no rim edge is crossed then going along them yields the desired result. If at some point we reach a rim edge f_1 with crossings, then, instead of f_1 , we take the edge e_1 to the first vertex before v_a and continue in counter-clockwise order for the rest of the circle. In the same manner, we construct a crossing-free path \mathcal{P}_2 starting at v_b and visiting all vertices on the second circle, potentially using a circle edge e_2 instead of a rim edge f_2 with crossings. Finally, we connect the end-vertices of \mathcal{P}_1 and \mathcal{P}_2 that are different from v_a and v_b (unless the respective path has only one vertex) by a lateral edge e . This produces a Hamiltonian path \mathcal{P} in \mathcal{D} with end-vertices v_a and v_b . Further, \mathcal{P} is crossing-free because e , e_1 , and e_2 are the only edges in \mathcal{P} that could have crossings. However, if e_1 and/or e_2 are present, then f_1 and/or f_2 partition the lateral face into up to three areas such that e is contained in one area A of them, and e_1 and e_2 are contained in two different connected components of the complement of A .



■ **Figure 14** Construction of a crossing-free Hamiltonian path between two given vertices v_a and v_b in a cylindrical drawing when v_a and v_b lie on (a) different circles or (b) the same circle. In both situations the case with a crossed rim edge is shown.

Let next v_a and v_b lie on the same circle C_1 and assume that there is at least one vertex on the other circle C_2 . See Figure 14(b) for an illustration. By Lemma 3.5, there exists a path \mathcal{P}_2 consisting of completely uncrossed rim edges that visits all vertices of C_2 . For connecting the remaining vertices, assume first that all rim edges on C_1 are completely uncrossed. Let then \mathcal{P}_1 be the path of completely uncrossed rim edges starting at v_a and visiting all vertices in clockwise order on C_1 until one vertex before v_b . Accordingly, let \mathcal{P}_3 be the path of completely uncrossed rim edges starting at v_b and visiting all vertices in clockwise order on C_1 until one vertex before v_a . By this, \mathcal{P}_1 and \mathcal{P}_3 cover all vertices of C_1 . In the remaining case, when there exists a unique rim edge f_1 on C_1 with crossings,

we assume without loss of generality that f_1 lies between v_a and v_b in clockwise direction along C_1 . Let then \mathcal{P}_1 be the completely uncrossed path starting at v_a and visiting all vertices in clockwise order on C_1 until the first end-vertex of f_1 . For the path \mathcal{P}_3 , we start at v_b and visit all vertices in clockwise order on C_1 via completely uncrossed rim edges until the last vertex v before v_a . If v_b is the second end-vertex of f_1 , then \mathcal{P}_1 and \mathcal{P}_3 already cover all vertices of C_1 . Otherwise, we extend \mathcal{P}_3 by the edge e_1 from v to the last vertex before v_b in clockwise order on C_1 and continue from there in counter-clockwise order along C_1 , again via completely uncrossed rim edges, until we reach the second end-vertex of f_1 . We then connect the three paths \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 with two lateral edges e and e' to a Hamiltonian path \mathcal{P} with end-vertices v_a and v_b . In particular, there are two choices on how to connect the end-vertices of \mathcal{P}_1 and \mathcal{P}_3 that are different from v_a and v_b (unless the respective path has only one vertex) with the end-vertices of \mathcal{P}_2 (unless \mathcal{P}_2 has only one vertex, but then the unique choice is crossing-free). At least one of those choices is a non-crossing edge pair e and e' because there can be at most one crossing induced by any 4-tuple of vertices. Consequently, the only potential crossings in \mathcal{P} are between the connection edges e and e' , and the non-rim edge e_1 , if it exists. However, since in that case f_1 splits the lateral face into two areas such that e_1 cannot enter the area containing e and e' , \mathcal{P} is again crossing-free.

Finally, if all vertices lie on one circle, then, by Observation 2.3, \mathcal{D} is strongly isomorphic to a 2-page-book drawing. Since every 2-page-book drawing is weakly isomorphic to an x -monotone drawing, the statement follows by Proposition 3.10. This completes the proof. ◀

Again, by Corollary 3.6, this implies that also Conjecture 1.1 holds for cylindrical drawings.

► **Corollary 3.16.** *Let \mathcal{D} be a cylindrical drawing of K_n with $n \geq 3$. Then \mathcal{D} contains a crossing-free Hamiltonian cycle.*

4 Relations between classes of simple drawings

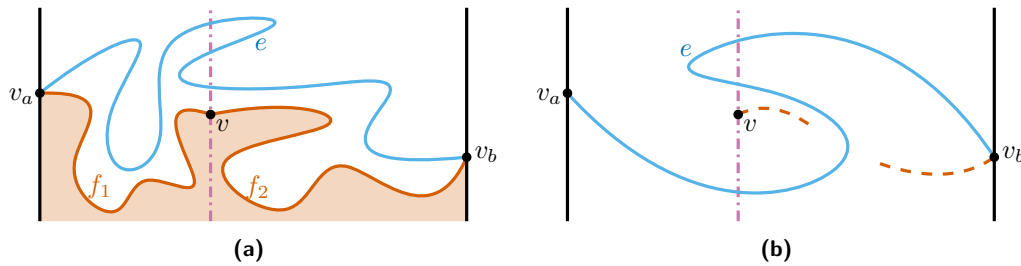
This section is devoted to analyzing inclusion relations between different classes of simple drawings, in particular, those that are marked darkorange in Figure 7.

4.1 X -bounded drawings

We start by showing that every x -bounded drawing \mathcal{D} of K_n is weakly isomorphic to an x -monotone drawing. This also follows from a result by Balko, Fulek, and Kynčl [13, Lemma 4.8]. However, since their proof relies on many other results in that paper, which makes the ideas behind it hard to grasp, we present a self-contained proof here. The main ingredient for our proof is given by the following lemma.

► **Lemma 4.1.** *Let $e = \{v_a, v_b\}$ be an edge and v a vertex with x -coordinate between v_a and v_b in an x -bounded drawing of K_n . Then e crosses the vertical line through v either above v or below v , at least once, but never on both sides.*

Proof. Recall from Section 3.2 that e splits the vertical strip between v_a and v_b into two sides, one above e and one below e . We assume, without loss of generality, that v lies below e and that v_a is left of v_b ; see Figure 15(a) for an illustration. Further, we consider the edges $f_1 = \{v_a, v\}$ and $f_2 = \{v, v_b\}$. By Observation 3.9, both f_1 and f_2 lie completely below e and, by x -boundedness, f_1 lies left of the vertical line through v and f_2 lies to its right. Hence, the union of f_1 and f_2 splits the vertical strip between v_a and v_b into two parts, an upper and a lower part, such that e is contained in the upper part and the vertical ray that starts in v and goes downwards is contained in the lower part. Consequently, e only crosses the vertical line through v above v , which it must cross at least once to connect v_a and v_b . ◀



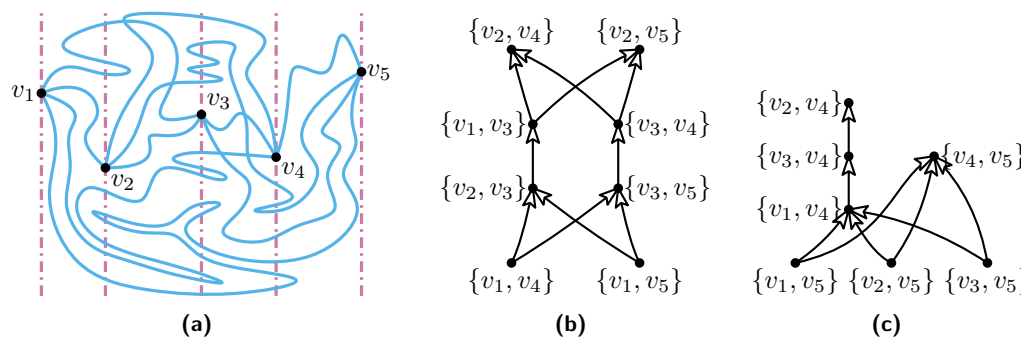
■ **Figure 15** (a) The edge e can only cross the vertical violet line through v on one side, above or below, because the other side lies in the orange area bounded by f_1 and f_2 . (b) The edge e crosses the vertical line through v on both sides; the dashed darkorange edge $f_2 = \{v, v_b\}$ cannot be inserted anymore within the vertical strip between v and v_b without crossing its incident edge e .

It is crucial for the proof that both edges f_1 and f_2 exist in the drawing. In the other direction, if the edge e crosses the vertical line through v on both sides, then at least one of those two edges cannot be added in an x -bounded way anymore; see Figure 15(b). Hence, Lemma 4.1 and subsequent results only hold for x -bounded drawings of complete graphs.

From here on the arguments are mostly of technical nature. We start by introducing, for each vertex v of an x -bounded drawing \mathcal{D} , a partial order $<_v$ on the set of edges in \mathcal{D} , defined by the following four conditions for $e <_v f$:

- e and f are incident to v , and e leaves v below f on the same side (left or right),
- e crosses the vertical line through v below v and f is incident to v ,
- f crosses the vertical line through v above v and e is incident to v , or
- the vertical line through v is crossed below v by e and above v by f .

For a fixed vertex v , any of these four conditions potentially induces an order between two edges. However, by Lemma 4.1, for each pair of edges at most one of the four conditions holds and the fourth condition is well-defined. Note that there is no relation between two edges e and f if either they both cross the vertical line through v on the same side of v or one lies completely to the left of v and the other lies completely to the right of v . Further, it can be verified that antisymmetry and transitivity are fulfilled. Since no edge gets related to itself, the conditions in fact induce a partial order. In Figure 16 we show two examples of such a partial order of edges at a vertex.



■ **Figure 16** (a) A realization of the twisted drawing \mathcal{T}_5 as a quite wiggly x -bounded drawing; the violet lines mark the bounds for the edges. (b) A Hasse diagram for the partial order $<_{v_3}$ of the drawing in (a). (c) A Hasse diagram for the partial order $<_{v_4}$ of the drawing in (a).

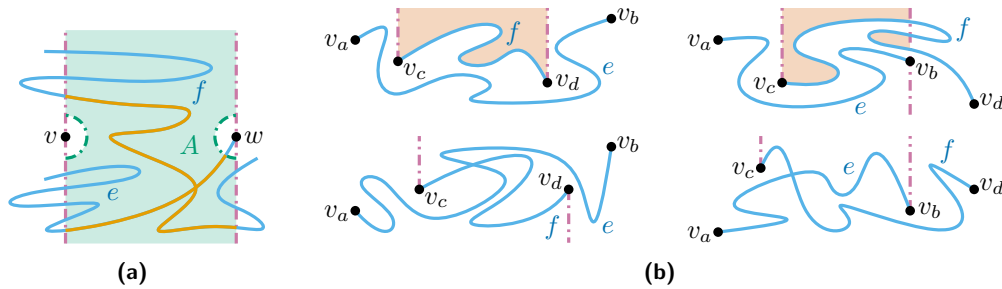
We next discuss how to determine all crossings of an x -bounded drawing of K_n by those partial orders. Recall that we also have a total order $<$ on the vertices from left to right. In particular, for $v < w$, the partial orders $<_v$ and $<_w$ determine, for two comparable edges, in which order from bottom to top they enter and leave, respectively, the vertical strip between v and w . See Figure 17(a) for an example illustration of the following observation.

► **Observation 4.2.** *Let \mathcal{D} be an x -bounded drawing of K_n . If, for two edges e and f and vertices $v < w$ in \mathcal{D} , the inequalities $e <_v f$ and $e >_w f$ hold, then e and f have a crossing in the vertical strip between v and w .*

Especially, if two edges are nested or linked, then they are always comparable at exactly two of their four end-vertices. Hence, using Observation 4.2 and extending Observation 3.9, we can classify in which pattern two edges e and f have to pass below or above each others end-vertices to form a crossing, depending on whether e and f are nested or linked; if they are separated, they cannot cross anyway. See Figure 17(b) for an illustration of some cases from the following observation.

► **Observation 4.3.** *Let \mathcal{D} be an x -bounded drawing of K_n with vertices v_1, \dots, v_n from left to right. Let $e = \{v_a, v_b\}$ and $f = \{v_c, v_d\}$ be two edges in \mathcal{D} with $a \leq c$ and, by convention, $a < b$ and $c < d$. Then e and f cross if and only if one of the following two conditions holds:*

- *e and f are nested, and $(e <_{v_c} f$ and $e >_{v_d} f)$ or $(e >_{v_c} f$ and $e <_{v_d} f)$; or*
- *e and f are linked, and $(e <_{v_c} f$ and $e >_{v_b} f)$ or $(e >_{v_c} f$ and $e <_{v_b} f)$.*



■ **Figure 17** (a) Illustration of Observation 4.2: The orange parts of e and f (between the last entry point from the left and the first exit point after that to the right) must cross within the seagreen area A (the vertical strip between v and w excluding a small ε -ball around v and w , each). (b) Illustrations of Observation 4.3: Two nested cases on the left and two linked cases on the right. In the bottom cases the edges cross by Observation 4.2. In the top cases the edges cannot cross because e cannot enter the darkorange areas above f without crossing f at least twice.

We stated the above observation for $a \leq c$ to also cover the case of incident edges, which cannot cross anyway. With this, we are ready to prove that every x -bounded drawing of K_n is weakly isomorphic to an x -monotone drawing. This is similar in spirit to a result shown by Fulek, Pelsmajer, Schaefer, and Štefankovič [19, Lemma 2.6]: Every x -bounded drawing can be made x -monotone without changing the parity of crossings between any pair of edges or changing the rotation around any vertex. While their result also holds for drawings of non-complete graphs, however, in their setting the drawings need not be simple. In particular, even if the initial x -bounded drawing is simple, the resulting x -monotone drawing after applying their transformation might not be simple anymore.

► **Theorem 4.4.** *Let \mathcal{D} be an x -bounded drawing of K_n with vertices v_1, \dots, v_n in that order from left to right. Then there exists an x -monotone drawing \mathcal{D}' that is weakly isomorphic to \mathcal{D} and has the same vertex order.*

Proof. We will construct an x -monotone drawing \mathcal{D}' that has the exact same set of crossing edge pairs as \mathcal{D} and the same vertex order v_1, \dots, v_n from left to right; see Figure 18 for an example illustration of the following steps.

We place the vertices v_1, \dots, v_n in that order on a horizontal line. For the edges, we consider the vertical strips between the vertices from left to right and, in each strip, add all edges simultaneously as follows. We define orders on where the edges enter the strip from the left as well as where they leave the strip to the right, from bottom to top in both cases. Subsequently, we connect all the respective entry and exit points by line segments.

In detail, we inductively consider the vertical strip between and including vertices v_i and v_{i+1} , for $i = 1, \dots, n - 1$. We can assume that all entry points from the left of edges crossing the vertical line through v_i below or above v_i are given by the exit points to the right in the vertical strip between vertices v_{i-1} and v_i . These entry points induce a linear order of those edges from bottom to top, which we denote by \prec_i^{\leftarrow} ; the order on the left boundary of the strip between v_i and v_{i+1} . For the first strip between vertices v_1 and v_2 , there are no such edges entering from the left, that is, the order is empty at this point. In any case, it remains to add the edges incident to v_i to this order.

For that, we insert all the edges incident to v_i that in \mathcal{D} leave v_i to the right into \prec_i^{\leftarrow} between the edges crossing the vertical line through v_i below v_i and those crossing the line above v_i . Moreover, we add the edges ordered according to the order \prec_{v_i} in \mathcal{D} , which is the same as the order given by the counter-clockwise rotation of these edges at v_i . Hence the resulting order \prec_i^{\leftarrow} agrees with the partial order \prec_{v_i} in \mathcal{D} .

Next we create an order \prec_i^{\rightarrow} ; the order on the right boundary of the strip between v_i and v_{i+1} . To this end, we split the edges into three groups, namely, the ones leaving the strip below v_{i+1} , the ones ending in v_{i+1} , and the ones leaving the strip above v_{i+1} . By Lemma 4.1, these three groups are well-defined. Also note that the edges in each of those three groups are in general not consecutive in \prec_i^{\leftarrow} . To obtain \prec_i^{\rightarrow} , we start with all edges leaving the strip below v_{i+1} , continue with all edges ending in v_{i+1} , and finish with all edges leaving the strip above v_{i+1} , in each of the three groups keeping the relative order between the edges as given by \prec_i^{\leftarrow} . With that, for two edges in different of those three groups, \prec_i^{\rightarrow} agrees with the partial order $\prec_{v_{i+1}}$ in \mathcal{D} .

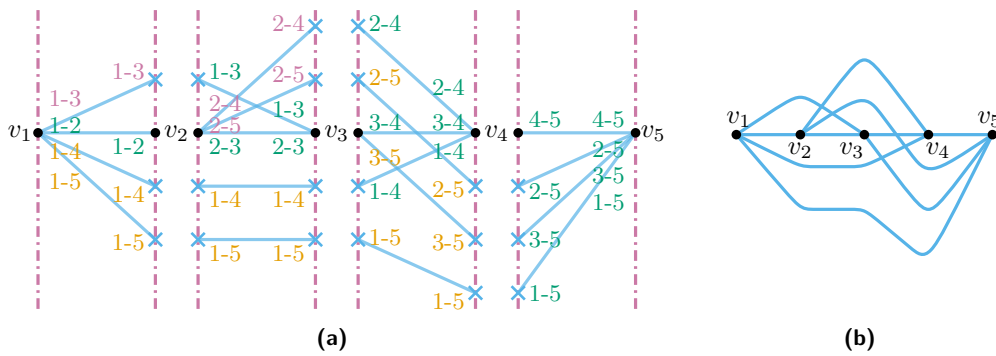


Figure 18 (a) The four steps of redrawing the x -bounded drawing from Figure 16(a) strip by strip into a weakly isomorphic x -monotone drawing. Note that each of the vertices v_2 to v_4 appears in two steps. The placement of the pairs “a-b”, for edges $\{v_a, v_b\}$, from bottom to top corresponds to the orders \prec_i^{\leftarrow} and \prec_i^{\rightarrow} on the left and right boundary of each strip, respectively. The colors of the pairs indicate the three groups, into which the edges are placed in \prec_i^{\rightarrow} : Orange for edges leaving the strip below v_{i+1} , seagreen for edges ending in v_{i+1} , and violet for edges leaving the strip above v_{i+1} . (b) The final result, with the edges smoothed at the transition points between the strips.

Finally, we mark the exit points of the edges on the vertical line through v_{i+1} , in the order given by \langle_i^{\rightarrow} from bottom to top, such that every edge passing below or above v_{i+1} gets its individual exit point, while all edges ending in v_{i+1} share the position of v_{i+1} as their exit point. For each edge, we connect the corresponding entry point on the left boundary of the strip with the exit point on the right boundary of the strip by a line segment. Then two of those line segments, belonging to edges e and f , respectively, cross in \mathcal{D}' if and only if the order of e and f changes between \langle_i^{\leftarrow} and \langle_i^{\rightarrow} . Without loss of generality, let $e \langle_i^{\leftarrow} f$ and $f \langle_i^{\rightarrow} e$. We next show, by applying Observation 4.2, that this crossing between the line segments belonging to e and f in \mathcal{D}' uniquely corresponds to a crossing of e and f in \mathcal{D} .

Since, within each of the three groups on the right boundary of the strip, we keep the relative order from the left boundary of the strip, e and f must have been placed into different groups on the right. Therefore, $f \langle_i^{\rightarrow} e$ implies that also $f \langle_{v_{i+1}} e$ holds.

Regarding the partial order \langle_{v_i} the edges e and f might be incomparable. However, there must be a vertex $w \leq v_i$ such that e and f are comparable with respect to \langle_w ; this is at least the case for the start-vertex of either e or f . Let $k \leq i$ be maximal such that e and f are comparable with respect to \langle_{v_k} . Hence, for all $k < j \leq i$, the edges e and f both cross the vertical line through v_j on the same side of v_j . Since \langle_j^{\leftarrow} agrees with $\langle_{j-1}^{\rightarrow}$ for edges crossing above or below v_j , and $\langle_{j-1}^{\rightarrow}$ agrees with $\langle_{j-1}^{\leftarrow}$ for edges placed in the same group on the right boundary of the strip, $e \langle_i^{\leftarrow} f$ implies that $e \langle_k^{\leftarrow} f$ holds. Consequently, since e and f are comparable with respect to \langle_{v_k} , which agrees with \langle_k^{\leftarrow} , we get $e \langle_{v_k} f$. Hence, by Observation 4.2, e and f cross in the vertical strip between vertices v_k and v_{i+1} in \mathcal{D} .

Moreover, e and f are comparable with respect to $\langle_{v_{i+1}}$. So any potential further crossing between e and f in \mathcal{D}' would correspond to a crossing between e and f in \mathcal{D} that lies to the right of the vertical line through v_{i+1} . This cannot exist because \mathcal{D} is simple.

It remains to argue that every crossing in \mathcal{D} also exists in \mathcal{D}' . By Observation 4.3, we know that every crossing in \mathcal{D} is in one-to-one correspondence with a change of the order of the involved non-incident edges e and f between two partial orders \langle_{v_i} and \langle_{v_j} with $i < j$, that is, at two of the end-vertices of e and f . This change implies that also the orders \langle_i^{\leftarrow} and \langle_j^{\rightarrow} change accordingly, which produces a crossing in the construction of \mathcal{D}' . ◀

In our construction, edges cross at the latest possible moment, that is, in the rightmost strip of the area given by Observation 4.2. Also, we implicitly use Lemma 4.1 all the time because the orders \langle_{v_i} would not be well defined otherwise. Hence, similar to Lemma 4.1, Theorem 4.4 does not hold for drawings of non-complete graphs. As an example, Figure 19(a) depicts an x -bounded drawing \mathcal{D}_b for which, as we argue below, no weakly isomorphic x -monotone drawing \mathcal{D}_m exists. That is, there is no x -monotone drawing with the same set of crossing edge pairs as in \mathcal{D}_b .

Assume for a contradiction that \mathcal{D}_m exists. Note that Gioan's theorem [22, Theorem 3.10] does not hold for drawings of non-complete graphs. In particular, \mathcal{D}_b and \mathcal{D}_m might not even have the same rotation system, and structures like clouds or fishes that exist in \mathcal{D}_b might not exist in \mathcal{D}_m and vice versa. So, as we are not aware of any simpler method to show this, we check all 360 possible orders, that is, permutations modulo reflection, of the 6 vertices along the x -axis and argue in each case that the order is not possible for \mathcal{D}_m . In the following we explain the main steps and ideas, which might be of value on their own, but we leave the details to the interested reader.

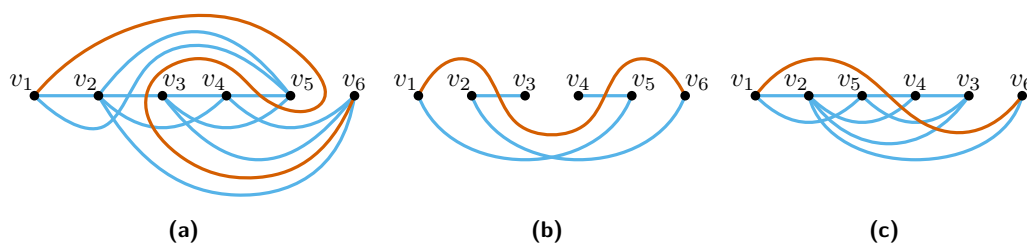
We start with some purely combinatorial arguments to reduce the number of orders that we have to check in detail. As in Observation 3.2, neither the first nor the last two vertices in the order are allowed to induce an edge with crossings. For example, the order

$(v_1, v_2, v_3, v_5, v_4, v_6)$ is not possible because the edge $\{v_4, v_6\}$ needs to cross the edge $\{v_3, v_5\}$. This kind of argument alone already leaves only 64 possible orders. More generally, two separated edges can never cross in an x -monotone drawing. Hence, for example, the order $(v_1, v_2, v_4, v_3, v_5, v_6)$ is not possible because the edge $\{v_2, v_4\}$ needs to cross the edge $\{v_3, v_5\}$. This further narrows it down to 40 potential orders. Furthermore, if an edge e crosses a triangle Δ an odd number of times, then one of the end-vertices of e must lie inside Δ and the other one outside. In particular, it is not possible that all three vertices of Δ lie between the end-vertices of e in the x -monotone order because then both end-vertices of e would definitely lie outside of Δ . For example, the triangle $\{v_2, v_4, v_5\}$ is crossed once by the edge $\{v_3, v_6\}$, which forbids the order $(v_1, v_3, v_2, v_4, v_5, v_6)$. This eliminates 19 more cases.

To rule out the remaining 21 cases we use Observation 4.3, which establishes a connection between crossings, and edges passing below and above each others end-vertices. This observation also holds for non-complete graphs in the case of x -monotone drawings; the graph being complete in the case of x -bounded drawings is only needed to have the orders $<_v$ well defined. As an example, we consider the order $(v_1, v_2, v_3, v_4, v_5, v_6)$; see Figure 19(b) for an illustration. We first add the edge $e = \{v_1, v_6\}$ such that, without loss of generality, v_2 lies below e . Then v_3 has to lie above e because $\{v_2, v_3\}$ crosses e , v_4 also has to lie above e because $\{v_3, v_4\}$ does not cross e , and v_5 has to lie below e again because $\{v_4, v_5\}$ crosses e . Further, the edge $\{v_2, v_6\}$ passes below v_4 and does not cross $\{v_4, v_5\}$, so it has to pass below v_5 as well. Finally, the edge $\{v_1, v_5\}$ has to pass below v_2 to cross $\{v_2, v_6\}$, but then it cannot cross $\{v_2, v_3\}$ anymore; a contradiction. Applying a similar reasoning for the other 20 cases then finishes the proof that there is no x -monotone drawing \mathcal{D}_m being weakly isomorphic to \mathcal{D}_b .

Let us remark that removing the edges $\{v_3, v_6\}$ and $\{v_4, v_6\}$ from \mathcal{D}_b yields a sub-drawing \mathcal{D}'_b for which a weakly isomorphic x -monotone drawing \mathcal{D}'_m exists. It is reached by changing the position of the edges $\{v_2, v_4\}$ and $\{v_3, v_5\}$ in the rotation of their end-vertices though; see Figure 19(c). This also emphasizes that for drawings of non-complete graphs two different rotation systems can produce the same set of crossing edge pairs.

Further, note that in \mathcal{D}_b only the three edges $\{v_1, v_3\}$, $\{v_1, v_4\}$, and $\{v_5, v_6\}$ are missing, which cannot be added in an x -bounded way anymore once $\{v_1, v_6\}$ is fixed; recall Figure 15(b). All three of them can easily be added to create a (general) simple drawing of K_6 though.



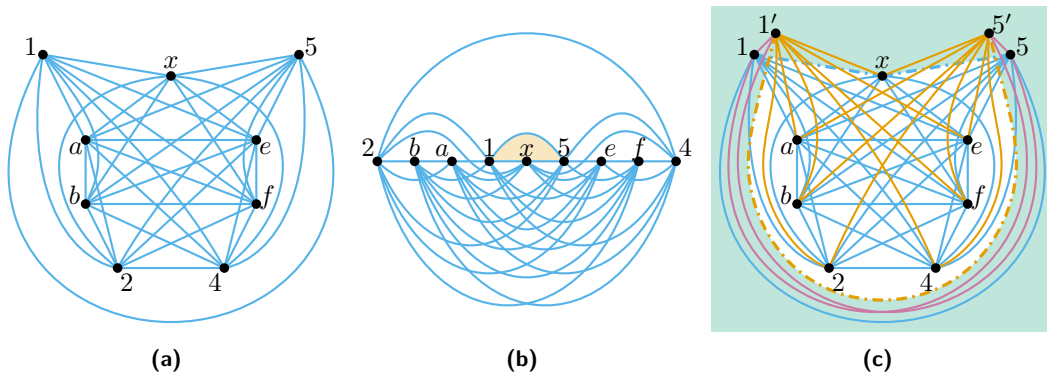
■ **Figure 19** (a) An x -bounded drawing \mathcal{D}_b of a non-complete graph that is not weakly isomorphic to any x -monotone drawing; the darkorange edge is x -bounded but not x -monotone. (b) An illustration showing that there cannot be any x -monotone drawing \mathcal{D}_m on the given vertex order being weakly isomorphic to \mathcal{D}_b . (c) An x -monotone drawing \mathcal{D}'_m that is weakly isomorphic to a sub-drawing of \mathcal{D}_b but has a different (partial) rotation system.

We conclude Section 4.1 with an example of a shellable drawing of K_{11} that is not weakly isomorphic to any x -bounded drawing. A similar result was already claimed by Balko, Fulek, and Kynčl [13, Theorem 4.9] for a shellable drawing \mathcal{D}_9 of K_9 ; see Figure 20(a). However, what they actually proved is that there exists no x -monotone drawing that is weakly

isomorphic to \mathcal{D}_9 and has the same unbounded cell as \mathcal{D}_9 . In particular, in Figure 20(b) we present an x -monotone drawing that is weakly isomorphic to \mathcal{D}_9 ; the cell corresponding to the original unbounded cell of \mathcal{D}_9 is shaded orange. Nevertheless, we can extend their example by two vertices to a simple drawing \mathcal{D}_{11} that actually has the claimed property; see Figure 20(c). In particular, we duplicate the vertices 1 and 5 within the original unbounded cell, similar to the construction in Figure 9. Clearly \mathcal{D}_{11} is still shellable. Moreover, if we remove the vertices 1 and 5 from \mathcal{D}_{11} , we get a drawing \mathcal{D}'_9 that is weakly isomorphic to \mathcal{D}_9 . Hence, by the result of Balko, Fulek, and Kynčl, there does not exist any x -monotone drawing that is weakly isomorphic to \mathcal{D}_{11} and has its unbounded cell in the union of the unbounded cells of \mathcal{D}_9 and \mathcal{D}'_9 ; shaded seagreen in Figure 20(c). On the other hand, this union is exactly the inside of a fish formed by parts of the edges $\{1, x\}$, $\{x, 5\}$, and $\{1', 5'\}$; dash dotted in Figure 20(c). Similar to c -monotone drawings, the unbounded face in any x -monotone drawing has to lie in the inside of every fish. Putting these two arguments together, we conclude that there is no x -monotone or, by Theorem 4.4, x -bounded drawing that is weakly isomorphic to \mathcal{D}_{11} .

Two more remarks on the above reasoning: First, in the x -monotone drawing given in Figure 20(b), the described vertex duplication would destroy x -monotonicity because, for example, we would have to place $1'$ inside the orange cell and route the edge $\{1', 5\}$ around vertex 1. And second, we can repeat the duplication process on \mathcal{D}_{11} to get arbitrarily large shellable drawings that are not weakly isomorphic to any x -bounded drawing.

► **Observation 4.5.** *For all $n \geq 11$ there exist shellable drawings of K_n that are not weakly isomorphic to any x -bounded drawing.*



■ **Figure 20** (a) The original shellable drawing \mathcal{D}_9 from Balko, Fulek, and Kynčl [13, Figure 21] with the vertices labeled as in their paper. (b) An x -monotone drawing that is weakly isomorphic to \mathcal{D}_9 . (c) The extended shellable drawing \mathcal{D}_{11} , which is not weakly isomorphic to any x -bounded drawing anymore. The new edges are orange if they are part of \mathcal{D}'_9 and purple otherwise.

4.2 Cylindrical drawings and c -monotone drawings

Finally, we turn our attention to the different types of cylindrical drawings. Specifically, we will show that every cylindrical drawing is weakly isomorphic to a c -monotone drawing and that every strongly cylindrical drawing is weakly isomorphic to a strongly c -monotone drawing. Let us emphasize that all results in this section hold for drawings of arbitrary graphs, not just complete graphs. Our first step is to prove that we can redraw a cylindrical drawing \mathcal{D} , without changing any crossing properties, such that all edges are c -monotone with respect to the common center O of the two circles. To this end we define, similar to the

winding number of closed curves in complex analysis, the *continuous winding number* ω_e of an edge e as the overall portion of times, as a real number, that e completely travels around O in counter-clockwise direction. To fix the sign of ω_e , we orient all lateral edges of \mathcal{D} from the outer to the inner circle and we give all circle edges an arbitrary but fixed orientation. A negative value of ω_e then means that e travels in clockwise direction around O . To ensure that ω_e is well-defined for all edges e , we can assume, without loss of generality, that no inner circle edge actually passes through O . We can give a bound on ω_e , simultaneously for every edge e in \mathcal{D} .

► **Lemma 4.6.** *Let \mathcal{D} be a cylindrical drawing. Then there exists a cylindrical drawing \mathcal{D}' that is strongly isomorphic to \mathcal{D} and has $|\omega_e| < 1$ for every edge e in \mathcal{D}' .*

Proof. Observe that $|\omega_e| < 1$ holds anyway for every circle edge e because otherwise e would have to cross itself. Let further e_0 and e_1 be two lateral edges for which $\omega_{e_0} = \min_{e \in E}(\omega_e)$ and $\omega_{e_1} = \max_{e \in E}(\omega_e)$ holds, where E is the set of all lateral edges in \mathcal{D} . Observe that $\omega_{e_1} - \omega_{e_0} < 2$ because otherwise e_0 and e_1 would have to cross each other at least twice.

So there exists a homeomorphism of the plane, rotating the outer circle appropriately, that transforms \mathcal{D} into a cylindrical drawing \mathcal{D}' with $|\omega_e| < 1$ for every edge e in \mathcal{D}' . ◀

In other words, we can transform every cylindrical drawing \mathcal{D} such that every edge e of \mathcal{D} travels less than one round around O , which is a necessary condition for e to be c-monotone with respect to O . To further transform \mathcal{D} into a c-monotone drawing, it basically remains to “stretch” all its edges.

► **Proposition 4.7.** *Let \mathcal{D} be a cylindrical drawing. Then there exists a c-monotone drawing \mathcal{D}' that is weakly isomorphic to \mathcal{D} .*

Proof. We argue that all crossings in \mathcal{D} are combinatorially determined by the position of the vertices on the two circles, by the choice of whether an edge lies in the inner, lateral, or outer face, and by the direction an edge travels around O .

By Observation 2.3 the sub-drawings induced by each of the two circles are weakly isomorphic to 2-page-book drawings. So the crossings between two circle edges on the same circle are determined by Observation 2.1. Further, the crossings between circle edges and lateral edges are determined by Observation 3.3. This observation also implies that there are no crossings between two circle edges from different circles. Finally, two lateral edges e and f do *not* cross if and only if $0 \leq \delta + \omega_f - \omega_e \leq 1$, where δ is the fraction of the outer circle from the end-vertex of e to the end-vertex of f in counter-clockwise direction.

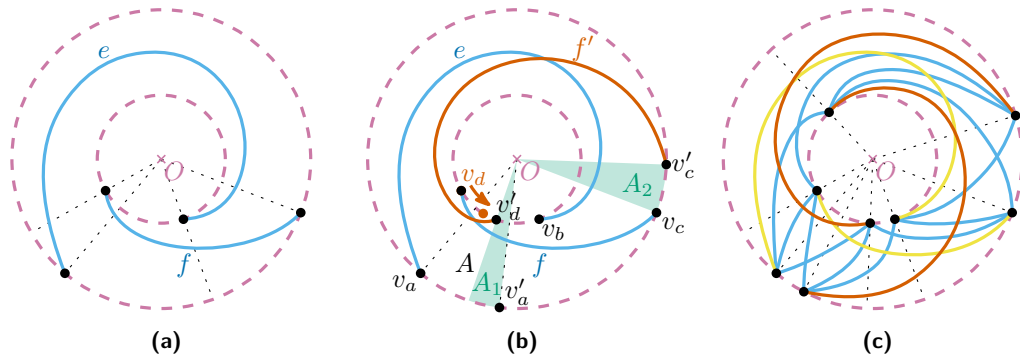
Since we can also assume, by Lemma 4.6, that $|\omega_e| < 1$ holds for every edge e in \mathcal{D} , we can easily redraw all edges such that they are c-monotone with respect to O and without changing, for any edge e , its ω_e value, the positions of its end-vertices, or the face in which e is drawn. By the above arguments, this results in a c-monotone drawing \mathcal{D}' that is weakly isomorphic to \mathcal{D} . ◀

To show that every strongly cylindrical drawing \mathcal{D} is weakly isomorphic to a strongly c-monotone drawing, we first focus on the lateral edges and assume, by Proposition 4.7, that \mathcal{D} is c-monotone. Especially, we use the notion of *wedges* and *covering the plane* from c-monotone drawings also for \mathcal{D} . Note that two incident lateral edges can never cover the plane because they otherwise would have to cross each other. However, two non-incident lateral edges e and f might cover the plane. In that case the signs of ω_e and ω_f must be the same though because otherwise e and f would cross each other twice. We call such a pair of lateral edges with negative signs a *clockwise double-spiral* and with positive signs a *counter-clockwise double-spiral*; see Figure 21(a) for an example.

► **Lemma 4.8.** *Let \mathcal{D} be a cylindrical drawing. Then there exists another cylindrical drawing \mathcal{D}' that is weakly isomorphic to \mathcal{D} and contains no double-spirals.*

Proof. We will remove one double-spiral at a time by moving around vertices on the inner circle, via a homeomorphism of the plane, without changing their order on the circle. By Proposition 4.7 and its proof, we can assume that the initial drawing \mathcal{D} is c-monotone and we can make each drawing in the process, that results from moving some vertices, again c-monotone without changing the vertex positions.

Let $e = \{v_a, v_b\}$ and $f = \{v_c, v_d\}$ form a clockwise double-spiral with v_a and v_c on the outer circle; see Figure 21(b) for an illustration. Let further v'_a be the neighboring vertex of v_a on the outer circle in counter-clockwise direction. Then we move vertex v_d in counter-clockwise direction on the inner circle out of the wedge Λ_e and into the wedge A between v_a and v'_a . This removes the double-spiral formed by e and f . To keep the circular order of vertices and therefore weak isomorphism, we also move each vertex between the old and the new position of v_d on the inner circle in counter-clockwise direction into the wedge A .



■ **Figure 21** (a) Two edges e and f forming a clockwise double-spiral. (b) Moving vertex v_d , and potentially other vertices v'_d , to resolve the clockwise double-spiral formed by e and f . This cannot create any new counter-clockwise double-spirals. (c) A clockwise (yellow) and a counter-clockwise (darkorange) double-spiral in a cylindrical drawing of K_8 ; circle edges are omitted for convenience.

It remains to show that no new double-spirals are created in the process. Since we only move vertices on the inner circle in counter-clockwise direction, we cannot create any new clockwise double-spirals in the process. Moreover, every new counter-clockwise double-spiral must involve a vertex v'_d that is moved. So let $f' = \{v'_c, v'_d\}$ be an edge with $\omega_{f'} > 0$ after v'_d is moved. By simplicity with the edge f , v'_c has to lie between v_c and v_a , v_c excluded, in counter-clockwise direction on the outer circle. Hence there exist wedges A_1 between v'_d and v'_a and A_2 between v_c and v'_c that lie in the closed complement of $\Lambda_{f'}$ and have disjoint interiors. In particular, every lateral edge e' with $\omega_{e'} > 0$ passing through A_1 in counter-clockwise direction, to potentially form a double-spiral with f' , must start on the outer circle after v'_c but at latest at vertex v_a and, to keep simplicity with the edge e , end before vertex v_b on the inner circle. Since v_c lies in Λ_e and, therefore, between v_b and v'_c in counter-clockwise direction around O , e' cannot pass through A_2 . Consequently, f' cannot be part of any counter-clockwise double-spiral.

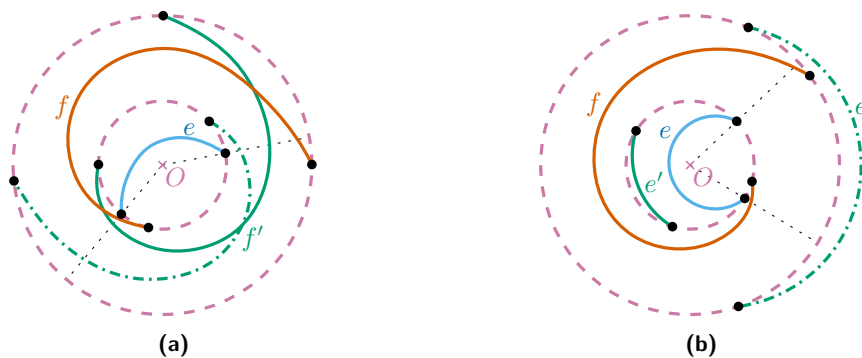
We proceed similarly to remove counter-clockwise double-spirals, moving vertices on the inner circle in clockwise direction. In each of those steps, we reduce the total number of double-spirals by at least one. So after finitely many steps we reach a weakly isomorphic cylindrical drawing \mathcal{D}' without double-spirals. ◀

We actually could have formulated Lemma 4.8 using the *strong* isomorphism. The *weak* isomorphism is only due to keeping the drawing c-monotone throughout the proof, which we do for convenience because it allows for a nice definition of double-spirals. Let us also remark that even a cylindrical drawing of K_n can contain a clockwise and a counter-clockwise double-spiral at the same time; see Figure 21(c).

Recall for the final proof of this paper that in a strongly cylindrical drawing, by definition, all circle edges must lie in the inner or outer face. Furthermore, we say that a circle edge e covers another circle edge e' , on the same or the other circle, if both end-vertices of e' lie in Λ_e , that is, e and e' together could potentially cover the plane, depending on $\Lambda_{e'}$.

► **Theorem 4.9.** *Let \mathcal{D} be a strongly cylindrical drawing. Then there exists a strongly c-monotone drawing \mathcal{D}' that is weakly isomorphic to \mathcal{D} .*

Proof. We can assume, by Proposition 4.7 and Lemma 4.8, that \mathcal{D} is c-monotone and contains no double-spirals. Hence the sub-drawing \mathcal{D}_ℓ of lateral edges is already strongly c-monotone. We construct \mathcal{D}' from \mathcal{D}_ℓ by suitably adding all circle edges to it from scratch. Since the face for each circle edge is fixed by being strongly cylindrical and the vertex positions are fixed by \mathcal{D}_ℓ , all crossings are already determined up to parity. The remaining task is to determine Λ_e , for each circle edge e , by choosing a direction around O , clockwise or counter-clockwise, without violating strong c-monotonicity or simplicity of \mathcal{D}' . Because two circle edges crossing each other twice would also mean that they together cover the plane, we can restrict our attention to strong c-monotonicity.



■ **Figure 22** (a) The lateral edge f excludes one direction for e . Another lateral edge f' cannot exclude the second direction for e because otherwise f and f' would either cross each other twice (solid f') or form a double-spiral (dash dotted f'). (b) The lateral edge f also excludes one direction for all circle edges e' covered by e ; the dash dotted example for e' is not possible.

We first consider possible interactions between Λ_e and Λ_f , for a circle edge e and a lateral edge f , once we fix one of the two directions for e , that is, whether e together with f might cover the plane. Assume there exist lateral edges f and f' such that one choice for the direction of e leads to e and f covering the plane and the other choice leads to e and f' covering the plane. Then f and f' would either cross each other twice or form a double-spiral, a contradiction in any case; see Figure 22(a) for an example. Hence, with respect to all lateral edges, the edge e has at least one of its two potential directions available.

Furthermore, if one direction of e is excluded by a lateral edge f , then f also excludes the respective direction for all circle edges that are covered by e , once e is drawn in its unique available direction; see Figure 22(b) for an example. That is, two circle edges e and e' that each only have one direction available will not cover the plane.

Finally, if there are circle edges with both directions available, we choose some ray r starting at O and use, for all those remaining edges, the direction such that they do not cross r . Consequently, no pair of those edges can cover the plane. Furthermore, let e be a circle edge that was forced into one direction by a lateral edge f and let e' be a circle edge that had both directions available. Then e and e' cannot cover the plane because otherwise also f and e' would cover the plane, a contradiction to e' having both directions available. Hence this produces a strongly c -monotone drawing \mathcal{D}' which is weakly isomorphic to \mathcal{D} . ◀

It is essential for the proof of Theorem 4.9 that the initial drawing \mathcal{D} is *strongly* cylindrical, because when a circle edge is drawn in the lateral face, then its direction around O is fixed from the start. In particular, we made use of that with the pretzel structure in Figure 12(b), where we constructed a cylindrical drawing that is not strongly c -monotone.

5 Conclusion

In this paper, we extended Rafla's Hamiltonian cycle conjecture for simple drawings of K_n (Conjecture 1.1) to a conjecture about Hamiltonian paths between given pairs of end-vertices (Conjecture 1.2). We gave an overview on known sub-classes of simple drawings (Section 2) and analyzed containment relations between them (Section 4). Moreover, we proved our new conjecture for several of those drawing classes, in particular, strongly c -monotone drawings and cylindrical drawings (Section 3.2).

A next goal is to extend our results to more classes of simple drawings, especially, generalized twisted drawings on an even number of vertices. From there we further aim for c -monotone drawings and crossing maximal drawings. The former are of interest due to the feature we mentioned after their definition in Section 2. The latter are of interest because they potentially contain only few plane sub-drawings.

Another intriguing question is to figure out the essential reason why Conjecture 1.1 should be true in general for simple drawings, while it is not true anymore for star-simple drawings. Moreover, it would be interesting to know whether the relation between Conjectures 1.1 and 1.2 (Theorem 3.7) can be formulated more specific for a single drawing instead of a whole set of drawings and whether Conjecture 1.1 also implies Conjecture 1.2. We remark, however, that even if Conjecture 1.2 is strictly stronger in the sense that Conjecture 1.1 does not imply Conjecture 1.2, it could potentially be easier to prove. Conversely, if Conjecture 1.2 should turn out to be false, this might give some insight on Rafla's original conjecture as well.

Finally, regarding relations between classes of simple drawings, we would like to know whether there is an inclusion relation between the rather combinatorially defined class of g -convex drawings and any of the more topologically defined classes like c -monotone drawings.

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