Bicolored Order Types

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Abstract

In their seminal work on Multidimensional Sorting, Goodman and Pollack introduced the so-called order type, which for each ordered triple of a point set in the plane gives its orientation, clockwise or counterclockwise. This information is sufficient to solve many problems from discrete geometry where properties of point sets do not depend on the exact coordinates of the points but only on their relative positions. Goodman and Pollack showed that an efficient way to store an order type in a matrix λ of quadratic size (w.r.t. the number of points) is to count for every oriented line spanned by two points of the set how many of the remaining points lie to the left of this line.

We generalize the concept of order types to bicolored point sets (every point has one of two colors). The bicolored order type contains the orientation of each bicolored triple of points, while no information is stored for monochromatic triples. Similar to the uncolored case, we store the number of blue points that are to the left of an oriented line spanned by two red points or by one red and one blue point in λ_B. Analogously the number of red points is stored in λ_R. As a main result, we show that the equivalence of the information contained in the orientation of all bicolored point triples and the two matrices λ_B and λ_R also holds in the colored case. This is remarkable, as in general the bicolored order type does not even contain sufficient information to determine all extreme points (points on the boundary of the convex hull of the point set).

We then show that the information of a bicolored order type is sufficient to determine whether the two color classes can be linearly separated and how one color class can be sorted around a point of the other color class. Moreover, knowing the bicolored order type of a point set suffices to find bicolored plane perfect matchings or to compute the number of crossings of the complete bipartite graph drawn on a bicolored point set in quadratic time.

Keywords and phrases Colored point sets, Order Types, Triple orientations, Complete bipartite graph

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1 Introduction

Many problems from discrete geometry are based on properties of point sets in the plane that do not depend on the exact coordinates of the points but only on their relative positions. In their seminal work on Multidimensional Sorting [6], Goodman and Pollack introduced the so-called order type, which for each ordered triple of a point set in the plane gives its orientation, clockwise, counterclockwise, or collinear. From a more global view, order types must fulfill some axioms that define abstract order types—see Knuth [9] for details on these axioms, where order types are called CC-systems. These axioms form one of the several axiom systems that define uniform acyclic rank-3 oriented matroids [3]. They all have their own applications and motivational aspects, and provide additional insight. Many of them can be generalized to higher ranks, while others are particulary natural in rank 3, like e.g. hyperlink sequences; see [4] and discussions therein for a short summary. However, motivated by the applications mentioned in the next paragraph, in this work we are mainly interested in order types which are realized as a set of points in the plane. For such a set S of points its order type can be stored in a matrix Λ, which is of cubic size with respect to the number of points of S. For every triple $p, q, s \in S$ we encode its orientation by $\Lambda(p, q, s) \in \{-1, 0, 1\}$ where "-1" means clockwise, "0" means collinear, and "$+1$" means counterclockwise.[1] An alternative way to code the order type is to use a matrix λ of quadratic size. For two points $p, q \in S$ the entry $\lambda(p, q)$ counts the number of points from $S \setminus \{p, q\}$ to the left of the oriented line through p and q. Goodman and Pollack [6] showed that the information contained in λ is the same as in Λ, and that the two matrices can be converted into each other in polynomial time.

Colored point sets have a long history in discrete and computational geometry—see the recent survey of Kano and Urrutia [7] for a nice collection of problems in this area. Consequently, we extend the concept of order types to bicolored order types. Let P be a set of at least 3 points in the plane in general position, that is, no three points lie on a common line. Using the symbol $\dot\cup$ for the disjoint union let $P = B \,\dot\cup\, R$ be partitioned into two disjoint sets B and R, $|B| = m$ and $|R| = n$, where the points $b_1, \ldots, b_m \in B$ are colored blue, and the points $r_1, \ldots, r_n \in R$ are colored red. An oriented line through two points $p, q \in P$, directed from p to q, will be denoted by ℓ_{pq}. For three points $p, q, s \in P$ the triple orientation $\Lambda(p, q, s)$ (clockwise or counterclockwise) is determined by considering the oriented line ℓ_{pq} and checking in which of the open half-planes defined by ℓ_{pq} (right or left) the point s lies. We denote the collection of the orientation of all bicolored point triples of P (triples that contain at least one blue and one red point) by $\Lambda(P)$ or simply by Λ if it is clear from the context which point set is considered. Labelled point sets where all triple orientations are the same belong to the same equivalence class, called the *bicolored order type* of P.

▶ **Definition 1.** *Let $\mathcal{P}$ be the set of all bicolored labelled point sets in general position in the plane consisting of m blue and n red points. The* bicolored order type *of size (m, n) is the equivalence class on $\mathcal{P}$ where two sets of $\mathcal{P}$ are equivalent if all bicolored triple orientations are the same.*

The orientation of monochromatic point triples is not encoded in Λ. Similar to λ for uncolored order types we define λ_B, λ_R to count the number of blue, respectively red, points to the left of an oriented line spanned by two points from P. More precisely, for any pair of red points $r_i, r_j \in R$ we count the number of blue points to the left of the oriented line $\ell_{r_i r_j}$ in $\lambda_B(r_i, r_j)$, and for any bicolored pair of points $b_i \in B$ and $r_j \in R$ we count the number of blue points to the left of the oriented line $\ell_{b_i r_j}$ in $\lambda_B(b_i, r_j)$ and the number of blue points to the left of the oriented line $\ell_{r_j b_i}$ in $\lambda_B(r_j, b_i)$. In the same way we count the number of red points to the left of an oriented line spanned by two blue points or by one blue and one red point in λ_R. See Figure 1 for an example.

In Section 2 we show as a central result that the information contained in λ_B and λ_R is equivalent to the information given by all bicolored triple orientations Λ. Moreover, given one of the two representations of a bicolored order type, the other can be derived in polynomial time. This result has to be seen in contrast to the fact that in general we cannot use the

[1] Note that Goodman and Pollack [6] define $\Lambda(p, q)$ as the set of points on the left of ℓ_{pq}. However, both definitions contain the equivalent information. For convenience, we directly use the triple orientation $\Lambda(p, q, s)$ of the point triple (p, q, s).

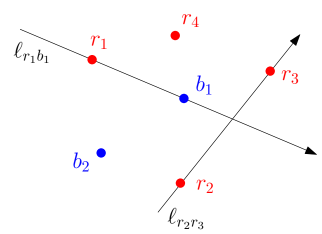


Figure 1 Counting points to the left of a line induced by the red point r_1 and the blue point b_1: $\lambda_B(r_1, b_1) = 0$ and $\lambda_R(r_1, b_1) = 2$. For the two red points r_2 and r_3, $\lambda_B(r_2, r_3) = 2$, while $\lambda_R(r_2, r_3)$ is not defined.

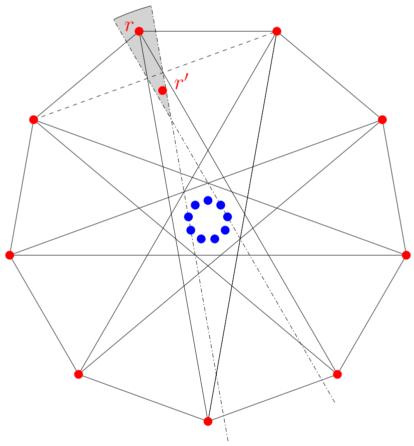


Figure 2 A point set where no point on the boundary of the convex hull can be determined. Point r can be be replaced by r' without changing the bicolored order type.

bicolored order type to determine all extreme points of a bicolored point set, see Figure 3 for an example. The reason is that the orientation of a monochromatic triple (the red point triple r_1, r_2, r_3 in Figure 3) is not encoded in the bicolored order type. Thus, r_2 can be extreme or not, without changing the bicolored order type of the set. Moreover, we might not even be able to determine any of the points on the boundary of the convex hull. An example is given in Figure 2. The boundary of the convex hull is built solely by red points, and this we can easily determine given the bicolored order type. However, the bicolored order type does not reveal any necessary information to determine whether a fixed red point lies on the boundary of the convex hull or not.

Using the equivalence between Λ and λ_B, λ_R we show that several tasks on bicolored point sets can be solved in polynomial time by using only the information contained in the bicolored order type. In Section 3 we show how to sort one color class around a point of the other color class, and how to determine whether the two color classes can be linearly separated.

In Section 4 we elaborate on how to determine crossings between bicolored edges. We use this to find bicolored plane perfect matchings for a given bicolored order type. For given λ_B and λ_R we show how to compute the number of crossings of the complete bipartite graph drawn on the represented bicolored point set in quadratic time. Finally, in Section 5 we generate all bicolored order types of small cardinality and give their numbers up to $m + n \leq 10$ and compare them to the number of uncolored order types.

2 Equivalence of (λ_B,λ_R) and Λ

Given the orientation of all triples Λ, in the uncolored case it is straightforward to compute the λ-matrix. Goodman and Pollack [6] showed that the opposite direction also holds, namely that the λ-matrix uniquely determines Λ. Given the λ-matrix, an extremal point is a point p for which $\lambda(p, q) = 0$ holds for some point $q \in P \setminus \{p\}$. To compute the triple orientations from the λ-matrix, we iteratively choose an extremal point p and remove it from the set after computing all triple orientations involving p. This is essentially done by sorting all

Figure 3 Two point sets with the same bicolored order type. We cannot determine whether r_2 lies on the boundary of the convex hull as the orientation of the red point triple (r_1, r_2, r_3) is unknown.

remaining points radial around the extremal point, using the information of λ to compare the order of two elements. Thus, this process is also called two-dimensional sorting, and more general multidimensional sorting in higher dimensions [6].

We show that a similar equivalence holds for bicolored order types. Again, given all bicolored triple orientations Λ, computing λ_B and λ_R is straightforward. So in the following, we will argue the inverse direction, which requires some more involved steps.

For bicolored point sets, a bicolored edge $\{b, r\}$ lies on the boundary of the convex hull if and only if either (1) $\lambda_B(b, r) = \lambda_R(b, r) = 0$ or (2) $\lambda_B(b, r) = m - 1$ and $\lambda_R(b, r) = n - 1$ holds. Then b and r are extreme points and can be found by testing all mn bicolored edges.

For monochromatic edges we can in general not determine whether they lie on the boundary of the convex hull, see Figure 3 for an example. The reason is that in λ_B and λ_R it is not encoded if points of the same color as the monochromatic edge lie on both sides of it. This also implies that we cannot always determine all extreme points.

However, if no bicolored edge on the boundary of the convex hull exists, we know that the boundary of the convex hull consists solely of points of one color. We thus can find a point of this color which is extremal to the set of the other color, that is, which lies outside of the convex hull of the points of the other color. We do this by inspecting all $\mathcal{O}(m^2 + n^2)$ monochromatic edges. For such an edge we check whether all points of the other color lie on one side of it. For example, for a blue edge $\ell_{b_i b_j}$ this is the case if either $\lambda_R(b_i, b_j) = 0$ or $\lambda_R(b_i, b_j) = n$. As obviously an extreme point (for example from a bicolored edge on the boundary of the convex hull) is also extremal to the set of the other color, we obtain the following observation.

▶ **Observation 2.** *For a given bicolored point set $P = B \mathbin{\dot\cup} R$ we can determine all points that are extreme w.r.t. the other color by using λ_B and λ_R in $\mathcal{O}(m^2 + n^2)$ steps.*

As we have seen, it is not always possible to determine whether a point is an extreme point using just λ_B and λ_R. We therefore extend the concept and make use of points that are not "dominated" by other points. We call such points *undominated*.

▶ **Definition 3.** *A red point $r \in R$ is undominated if it (1) lies outside of the convex hull of B, and (2) the wedge formed by the tangents from r to the convex hull of B that is opposite of B is empty of points of R. The symmetric definition holds for a blue point $b \in B$.*

▶ **Lemma 4.** *Given the matrices λ_B and λ_R and an undominated point $p \in B \mathbin{\dot\cup} R$, all bicolored triple orientations involving p can be determined in constant time per triple.*

Proof. Without loss of generality let $r \in R$ be an undominated red point. By definition, r lies outside the convex hull of B and the wedge between the two tangents of r to the convex hull of B that lies opposite of B is empty. We first compute the triple orientations of r and any two blue points; see Figure 4 for an illustration of the proof. Recall that for every $b \in B$

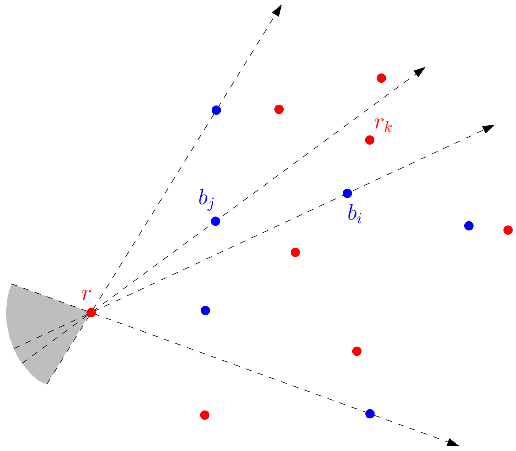


Figure 4 For an undominated point $r \in R$ the wedge marked in gray is empty of other points. All bicolored triple orientations involving r can be determined by using λ_B. In the drawing $\lambda_B(r, b_j) < \lambda_B(r, r_k) \leq \lambda_B(r, b_i)$ holds.

the number of blue points to the left of ℓ_{rb} is given by $\lambda_B(r, b)$. Thus, the rotational order of blue points around r can be read from λ_B, which yields the desired triple orientations.

$$\Lambda(r, b_i, b_j) = \Lambda(b_i, b_j, r) = \Lambda(b_j, r, b_i) = \begin{cases} +1, & \text{if } \lambda_B(r, b_i) > \lambda_B(r, b_j) \\ -1, & \text{otherwise.} \end{cases}$$

This implies the inverse triple orientations $\Lambda(b_j, b_i, r) = \Lambda(b_i, r, b_j) = \Lambda(r, b_j, b_i) = -\Lambda(r, b_i, b_j)$.

Next we consider the triple orientations involving r, another red point and one blue point. For every red point r_k, the number of blue points lying to the left of ℓ_{rr_k} is given by $\lambda_B(r, r_k)$. If a blue point b lies on the right side of ℓ_{rr_k}, then all blue points that lie to the left of ℓ_{rr_k} also lie to the left of ℓ_{rb}, as r is undominated. This is the case if and only if $\lambda_B(r, r_k) \leq \lambda_B(r, b)$. Thus we get

$$\Lambda(r, b, r_k) = \Lambda(b, r_k, r) = \Lambda(r_k, r, b) = \begin{cases} +1, & \text{if } \lambda_B(r, r_k) \leq \lambda_B(r, b) \\ -1, & \text{otherwise.} \end{cases}$$

Similar as before we have $\Lambda(r_k, b, r) = \Lambda(b, r, r_k) = \Lambda(r, r_k, b) = -\Lambda(r, b, r_k)$.

Observe that for every triple we only query two entries in λ_B. As claimed we can thus compute any triple orientation that involves r in constant time per triple. ◀

Next, we show that for a bicolored point set with a monochromatic, say red, convex hull we can always find an undominated red point by using only the information given in λ_B and λ_R. A red point r that is extremal with respect to B can easily be found via searching for another red point r_2 such that $\lambda_B(r, r_2) = 0$. Note that in that case both r and r_2 are extremal w.r.t. B, and that r_2 serves as a witness for r. The proof of the following Lemma describes how to additionally test whether r is undominated.

▶ **Lemma 5.** *Let $P = B \,\dot\cup\, R$ be a bicolored point set and let p be a point that is extreme with respect to the other color class. Then, in $O(m)$ steps for $p \in R$ (respectively $O(n)$ steps for $p \in B$) we can determine whether p is undominated.*

Proof. Assume without loss of generality that $r \in R$ is extremal with respect to B. From λ_B we can easily find the blue points b^1 ($\lambda_B(r, b^1) = 0$) and b^m ($\lambda_B(r, b^m) = m - 1$) of B that

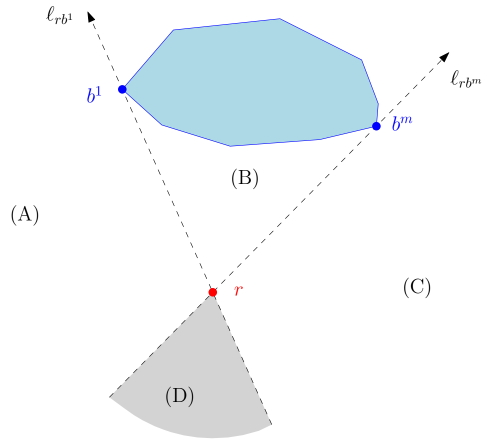


Figure 5 Test whether r is undominated. The tangents ℓ_{rb^1}, ℓ_{rb^m} define the four wedges (A), (B), (C) and (D).

define the left and the right tangent of r to the convex hull of B. If the wedge between these tangents through r that lies opposite of B—the wedge (D) marked in gray in Figure 5—is empty, then r is undominated. To check this, we first compute the number of red points in wedge (A); see again Figure 5. For every red point r_j we have $\lambda_B(r, r_j) = 0$ if and only if r_j lies in wedge (A). (Similarly, $\lambda_B(r, r_k) = m$ if and only if r_k lies in wedge (C)). Counting the corresponding entries in λ_B yields the number of red points in (A) (and also in (C)). Finally, $\lambda_R(r, b^1)$ gives the total number of red points in (A) and (D) together. Subtracting the number of red points in (A) from $\lambda_R(r, b^1)$ yields the number of red points in the wedge (D). If (D) is empty, then r is undominated by definition.

To find the two tangents through r and for counting the number of red points in (A) we have to inspect m entries of λ_B, all other steps need only constantly many values of λ_B and λ_R. Thus $O(m)$ steps are sufficient to determine whether r is undominated. ◀

▶ **Theorem 6.** *For a point set $P = B \,\dot\cup\, R$ the information contained in λ_B and λ_R is equivalent to the information given by all bicolored triple orientations Λ. Given one of the two representations of a bicolored order type, the other can be computed in polynomial time.*

Proof. Computing λ_B and λ_R from Λ can trivially be done in $\mathcal{O}(mn(m+n))$ steps. So assume we are given λ_B and λ_R and want to compute Λ. For any point set there always exist undominated points, as every extreme point is obviously an undominated point. Thus, by combining Observation 2 and Lemma 5 we can find an undominated point, say p, in time $\mathcal{O}(m^2+n^2)$. We do this by first computing all points that are extreme w.r.t. the other color ($\mathcal{O}(m^2+n^2)$ time) and then checking all these points for being dominated (in $\mathcal{O}(mn)$ time). The proof of Lemma 4 tells us how to compute all bicolored triple orientations involving p in time $\mathcal{O}(m^2+n^2)$. After this, we can remove p from the set and update the λ-matrices as follows. If w.l.o.g. we remove a red point $p = r_k$ then, for every triple (r_k, b_i, b_j), $i < j$, with $\Lambda(r_k, b_i, b_j) = 1$, r_k lies to the left of $\ell_{b_i b_j}$, so $\lambda_R(b_i, b_j)$ has to be decremented by 1 for each such triple. Similarly, if $\Lambda(r_k, b_i, b_j) = -1$, r_k lies to the right of $\ell_{b_i b_j}$ and thus $\lambda_R(b_j, b_i)$ is decremented by 1. For each triple consisting of r_k, a blue point b and another red point r_l, $r_l \neq r_k$, we also need to update λ_R. If $\Lambda(r_k, b, r_l) = 1$, then the value of $\lambda_R(b, r_l)$ is decremented by 1. Vice versa, in case $\Lambda(r_k, b, r_l) = -1$, $\lambda_R(r_l, b)$ gets decremented by 1. Moreover, the row and the column at index k are removed from both λ_B and λ_R as r_k is removed from the set. These updates can be done in total time $\mathcal{O}(m^2+n^2)$. We are left with λ-matrices of size $(m+n-1)\times(m+n-1)$. Doing this repeatedly $\mathcal{O}(m+n)$ times shows that all bicolored triple orientations Λ can be computed from λ_B and λ_R. The total number of steps needed is $\mathcal{O}(m^3+n^3)$. ◀

Given the equivalence of Theorem 6 it is natural to ask whether it suffices to use only bicolored edges for constructing Λ, meaning that the entries of both λ_B and λ_R are considered only at index pairs consisting of one red and one blue point. From Figure 15 we can see that this is in fact not the case. It is easy to verify that the two configurations given in Figure 15(a) and (b) have the same λ-matrices when restricting them to bicolored edges, namely

$$\lambda_B = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}\!\!\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \lambda_R = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}\!\!\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \end{array}.$$

Nevertheless, the triple orientations of (b_1,r_1,r_2), (b_1,r_1,r_3) and (b_1,r_2,r_3) are different.

3 Sorting and Separating

In the previous section we have shown that the information contained in the two λ-matrices on the one side and the Λ-matrix on the other side is equivalent. In this and the next section we will use this fact and show several combinatorial properties of bicolored point sets that can be derived from this information.

3.1 Sorting

In this section we show how to sort the set B of blue points clockwise around a given red point r, using the information in λ_B and Λ.

If r is extreme with respect to B, there exists a point $b\in B$ with $\lambda_B(r,b)=0$. The edge ℓ_{rb} is the left tangent from r to B and we choose b as the first point $b^1(r)$ in our sorting. The ordering of B around r can then be read from the matrix λ_B. A blue point is the i^{th} point $b^i(r)$ in the sorted order around r if exactly $i-1$ blue points lie to the left of $\ell_{rb^i(r)}$, that is, if $\lambda_B(r,b^i(r))=i-1$. Thus we can sort B around r by sorting the entries $\lambda_B(r,b_k)$ for all blue points $b_k\in B$ with a standard sorting algorithm.

If r lies in the interior of the convex hull of B then there is no unique first blue point in the order around r. Thus we first split the set B into two subsets, such that r is extreme w.r.t. both sets. Then we separately sort the blue points in each set around r, and finally combine the two sorted sets.

We start with an arbitrary blue pivot point $\tilde{b}$ and split the rest of B into two sets: the blue points that lie to the left of $\ell_{r\tilde{b}}$, and the blue points that lie to the right of this line. Let

$$B^+=\{b\in B\setminus\{\tilde{b}\}\mid \Lambda(r,\tilde{b},b)=+1\} \quad \text{and} \quad B^-=\{b\in B\setminus\{\tilde{b}\}\mid \Lambda(r,\tilde{b},b)=-1\}.$$

The line $\ell_{r\tilde{b}}$ defines the split between the first and the second part in the ordering around r, where $\tilde{b}$ will lie between them. Using Λ we compute $\lambda_{B^+}(r,b)$ for all $b\in B^+$ and analogously $\lambda_{B^-}(r,b)$ for all $b\in B^-$. Similar to the first case where r was extreme with respect to B we can sort the points in B^+ around r, and independently the points in B^- around r. Finally

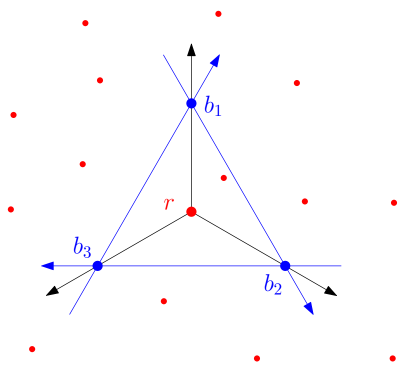


Figure 6 The order of $B = \{b_1, b_2, b_3\}$ around r does not follow in a direct way from λ_B and λ_R.

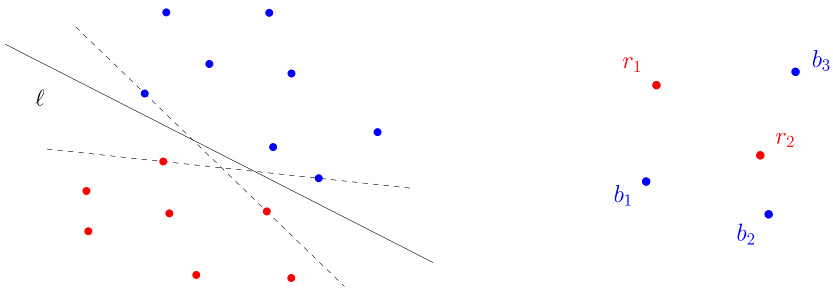


Figure 7 Left: ℓ linearly separates B and R. The dashed lines are the two bitangents of B and R. Right: B and R are not linearly separable.

we combine both sorted sets together with $\tilde{b}$ to the complete list of blue points sorted around r.

If r lies in the interior of $\text{conv}(B)$ it is not obvious how to solely use the information given by the entries of λ_B and λ_R to sort B around r. For an example see Figure 6. In this configuration we want to sort $B = \{b_1, b_2, b_3\}$ around r. The corresponding entries of λ_B are given by $\lambda_B(r, b_1) = \lambda_B(r, b_2) = \lambda_B(r, b_3) = 1$, but no direct conclusions can be drawn about the sorting of B around r. The information of λ_R also includes all red points that are not relevant in that case. This shows that for some applications it might be simpler to use the information of Λ, while for others using λ_R and λ_B might be easier. The equivalence shown in Theorem 6 allows us to take the perspective which seems to be more suitable for a problem at hand.

3.2 Linear Separation

Next, we show that the information encoded by λ_B and λ_R is sufficient to determine whether the two color classes B and R are linearly separable. If two sets of points are linearly separable then there exist two bitangents, c.f. Figure 7 (left). The bitangents can easily be determined via the λ-matrices: a line ℓ_{br} is a bitangent if and only if either

$$\lambda_B(b, r) = m - 1 \qquad \text{and} \qquad \lambda_R(b, r) = 0 \tag{1}$$

or

$$\lambda_B(b, r) = 0 \qquad \text{and} \qquad \lambda_R(b, r) = n - 1 \tag{2}$$

holds. Moreover, if there exists a line that satisfies Equation (1), then there also exists one that satisfies Equation (2) and vice versa. If, on the other hand, no pair (b, r) fulfills the conditions, then B and R are not linearly separable; see Figure 7 (right) for an example. The corresponding λ-matrices of this example are

$$\lambda_B = \begin{array}{c c} & \begin{array}{ccccc} b_1 & b_2 & b_3 & r_1 & r_2 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ r_1 \\ r_2 \end{array} & \begin{pmatrix} & & & 0 & 1 \\ & & & 1 & 1 \\ & & & 2 & 1 \\ 2 & 1 & 0 & & 1 \\ 1 & 1 & 1 & 2 & \end{pmatrix} \end{array} \quad \text{and} \quad \lambda_R = \begin{array}{c c} & \begin{array}{ccccc} b_1 & b_2 & b_3 & r_1 & r_2 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ r_1 \\ r_2 \end{array} & \begin{pmatrix} & 2 & 1 & 0 & 1 \\ 0 & & 2 & 0 & 1 \\ 1 & 0 & & 1 & 0 \\ 1 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix} \end{array}.$$

One can see that no tuple consisting of a blue and a red point satisfies condition (1) or (2). Hence, no bitangent exists in this point set.

4 Matchings and Crossings

In this section we concentrate on bicolored edges, that is, edges spanned by one red and one blue point. First we will show that using the bicolored order type allows us to construct a plane perfect red-blue matching. Then we consider crossings of bicolored edges and consider the crossing number of the complete bipartite graph $K_{m,n}$.

4.1 Bicolored Matchings

For this section we assume that our point set consists of as many blue points as red ones, that is, $|B| = |R| = n$.

▶ **Definition 7.** *If for a red point $r \in R$ and a blue point $b \in B$, $\lambda_B(r, b) = \lambda_R(r, b)$ holds, then we call ℓ_{rb} a balanced line.*

Note that since $|B| = |R|$, also the number of blue points to the right of the balanced line ℓ_{rb} equals the number of red points to the right of it. A special variant of a balanced line can be obtained from the 2-dimensional discrete version of the ham-sandwich cut. Here a line simultaneously bisects both the red and the blue points. That is, on each side of the line, there are at most half of the red and half of the blue points. Rotating a line around vertices and using the intermediate value theorem it can be shown that there always exists such a line that passes through one red and one blue point and that this line furthermore has the same number of red points as blue points to its left. Thus, this variant of the ham-sandwich cut provides a balanced line in our sense. In the following we consider a more general setting. Pach and Pinchasi [12], and later Orden, Ramos and Salazar [11], proved that there always exists a linear number of balanced lines.

▶ **Theorem 8.** *[12] For every bicolored point set with $|B| = |R| = n$ there exist at least n balanced lines.*

Using the concept of undominated points as introduced in Definition 3 we show that actually every undominated point is incident to a balanced line. This reveals more details about the configuration of bicolored point sets. However, the converse does not hold: there exist balanced lines which are defined by two dominated points.

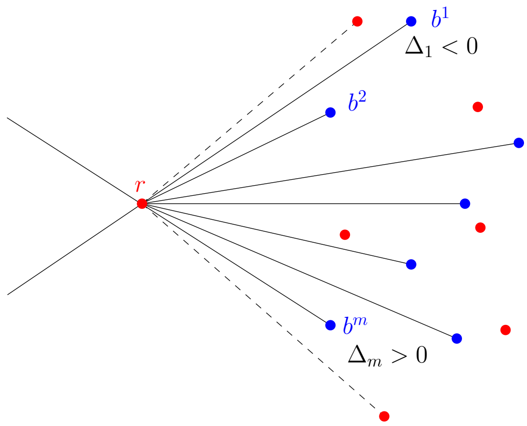


Figure 8 An undominated point r. The blue points $b^1, \ldots, b^n$ are sorted clockwise around it.

▶ **Theorem 9.** *Let P be a bicolored point set with $|B| = |R|$. For every red point $r \in R$ that is undominated, there exists a blue point $b \in B$ such that ℓ_{rb} is a balanced line. The analogous statements holds for undominated blue points.*

Proof. Let B be sorted clockwise around r and label the blue points $b^1, \ldots, b^m$ in this order; see Figure 8. Let $\Delta_i = \lambda_B(r, b^i) - \lambda_R(r, b^i)$ be the difference between the number of blue and the number of red points to the left of ℓ_{rb^i}. To prove the theorem we will show that there exists some i with $\Delta_i = 0$.

If there is no red point to the left of ℓ_{rb^1} then $\{r, b^1\}$ lies on the boundary of the convex hull of P and therefore ℓ_{rb^1} is a balanced line, and the theorem follows. The same holds if there is no red point to the right of ℓ_{rb^m}. So for the remainder of the proof we can assume that there is at least one red point to the left of ℓ_{rb^1} and at least one red point to the right of ℓ_{rb^m}, thus $\Delta_1 < 0$ and $\Delta_m > 0$. Note that these red points are disjoint, as r is undominated.

Next consider the sequence $\Delta_1, \ldots, \Delta_m$. Since $\Delta_1 < 0$ and $\Delta_m > 0$, the values of this sequence have to increase at some point. $\Delta_{i+1} > \Delta_i$ can only be true if no red point lies in the wedge b^i, r, b^{i+1}. Therefore the sequence increases by at most 1 in each step. Since it starts with $\Delta_1 < 0$ and ends with $\Delta_m > 0$, it needs to pass $\Delta_i = 0$ for some $i \in 1, \ldots, m$. ◀

The existence of a bicolored balanced line can be used to construct a plane perfect matching of bicolored edges by a divide-and-conquer algorithm. We successively find a balanced line ℓ_{rb}, add it to the matching, and split the rest of the point set into two subsets, where one contains all points to the left of ℓ_{rb}, and the other one contains all points to the right of ℓ_{rb}. Since the subsets are linearly separated, no two edges of the matchings from different iterations intersect, and thus we get a plane perfect matching. Finding a balanced line can be done using the information from λ_B and λ_R, and splitting the remaining points into two subsets can be done using Λ. Hence, we obtain the following result.

▶ **Corollary 10.** *The information contained in the bicolored order type of a set P with $|B| = |R|$ is sufficient to find a plane perfect matching of bicolored edges of P.*

Note that not every plane perfect matching with bicolored edges contains an edge that defines a balanced line. See Figure 9 for a class of matchings, so-called *lens shutters*, that do not contain any balanced line. In the next section we will see how to find crossings between bicolored edges, which can then also be used to check whether a given set of bicolored edges constitutes a plane perfect matching.

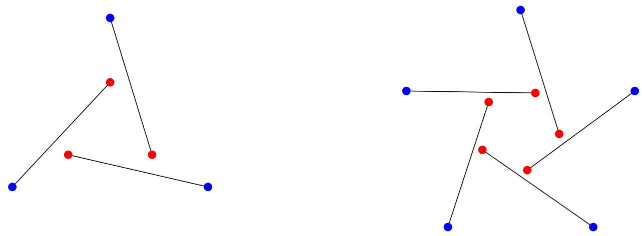
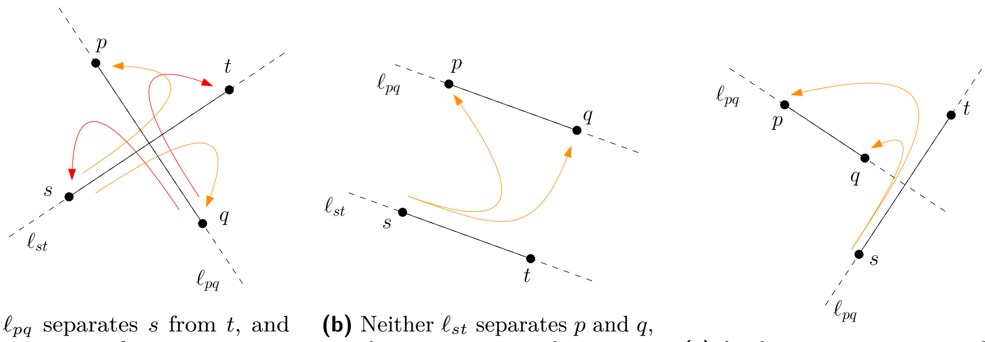


Figure 9 The *lens shutter*: two examples of matchings where no edge induces a balanced line.



(a) ℓ_{pq} separates s from t, and ℓ_{st} separates p from q.

(b) Neither ℓ_{st} separates p and q, nor ℓ_{pq} separates s and t.

(c) ℓ_{st} does not separate p and q.

Figure 10 Three combinatorially different configurations of two edges.

4.2 Crossings

To minimize the number of crossings in drawings of the complete bipartite graph has been a topic of intensive research in the last decades, see for example the survey by Schaefer [13]. In 1954, Zarankiewicz [16] conjectured the minimum number of crossings in any drawing of $K_{m,n}$ to be

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

and he constructed a straight-line drawing of the complete biparite graph which induces that many crossings. The conjecture has been verified for graphs of size (m,n) with $m \leq 6$ [8], and for $K_{7,7}$ and $K_{7,9}$ [15]. It is still open for $K_{7,11}$, $K_{9,9}$ and $K_{11,11}$. In this chapter, we present a formula to compute the number of crossings of a rectilinear drawing of $K_{m,n}$ on a given bicolored point set, using the matrices λ_B and λ_R. Our approach is significantly faster than simply testing all 4-tuples of points for a crossing.

Two edges $\{p,q\}$ and $\{s,t\}$ cross in their interior if and only if each of the lines ℓ_{pq} and ℓ_{st} separate the end points of the other edge, see Figure 10(a). Thus, to determine whether $\{p,q\}$ intersects $\{s,t\}$ we can use the triple orientations of their end points p,q,s, and t. More precisely, $\{p,q\}$ intersects $\{s,t\}$ if and only if (1) $\Lambda(p,q,s) \neq \Lambda(p,q,t)$ and (2) $\Lambda(s,t,p) \neq \Lambda(s,t,q)$ holds. Thus, checking four triple orientations is sufficient to determine crossings. In Figure 10(b) and (c) we depict configurations where one or both conditions do not hold, and therefore no intersection exist.

The concept of determining crossings between segments via triple orientations also works for colored points, as long as the needed triple orientations are defined. The case we are mainly interested in are crossings between two bicolored edges (b_1,r_1) and (b_2,r_2). In that case we have to check $\Lambda(b_1,r_1,b_2) \neq \Lambda(b_1,r_1,r_2)$ and $\Lambda(b_2,r_2,b_1) \neq \Lambda(b_2,r_2,r_1)$. All required triples are bicolored, and thus crossings of bicolored edges are well defined by a bicolored order type.

In contrast to the previous observation, it is in general not possible to detect an intersection by using the bicolored order type if one edge is bicolored, and the other edge is monochromatic. To see this, consider the two edges (b_1, b_2) and (b_3, r_1) and observe that we would need to check the triple orientation $\Lambda(b_1, b_2, b_3)$, which is not defined in the bicolored order type. The same is true if we consider two monochromatic edges that have the same color. But for a monochromatic red edge and a monochromatic blue edge all required triple orientations are again bicolored, and thus crossings can be detected.

4.2.1 Rectilinear Crossing Number

In this section we consider the rectilinear crossing number $\overline{\mathrm{cr}}(P)$ of the complete bipartite graph on $P = B \dot\cup R$. We will derive a formula for $\overline{\mathrm{cr}}(P)$ in a way similar to the formula for the rectilinear crossing number of the complete graph presented by Lovász, Vesztergombi, Wagner, and Welzl [10] and also independently by Ábrego and Fernández-Merchant [1]. Their basic idea is to compute the number of labeled 4-tuples of points in a set S of n' points and to compare it with the number of 4-tuples in convex and concave position. They define a *j-edge* as an ordered pair of points p, q such that exactly j points of the given point set lie on one side of ℓ_{pq}, and the remaining $(n' - 2 - j)$ points on the other side. Let e_j denote the number of j-edges. Then, every j-edge can be extended to a labeled 4-tuple of points p, q, s, t by adding one point s from the left of ℓ_{pq}, and one point t from the right. Counting all such 4-tuples sums up to $\sum_{j=0}^{n'-2} e_j\, j\, (n' - 2 - j)$. In that way every 4-tuple of points in convex position is counted 4 times (4 oriented edges separate the other two points), while every 4-tuple of points in non-convex position is counted 6 times (6 oriented edges separate the other two points). The total sum of 4-tuples is $\binom{n'}{4}$. Together with the fact that the number of crossings equals the number of convex 4-tuples this leads to the following formula for the rectilinear crossing number.

$$\overline{\mathrm{cr}}(S) = 3\binom{n'}{4} - \frac{1}{2}\sum_{j=0}^{n'-2} e_j\, j\, (n' - 2 - j)$$

For more details about this relation see [1, 10].

In a similar way we next derive a formula for the rectilinear crossing number $\overline{\mathrm{cr}}(P)$ of the complete bipartite graph on a point set $P = B \dot\cup R$. We do this by again counting 4-tuples of points, namely those consisting of two blue and two red points. More precisely, we count the edges that separate the other two points of a 4-tuple, and distinguish the types of separation according to the coloring. Given a 4-tuple, we call an edge a *separating edge* if the line going through the two endpoints of the edge separates the other two points of the 4-tuple. A separating edge can either be monochromatic, meaning it is defined by the two points of the same color and splits the two points of the other color, or it can be bicolored and splitting two points of different colors. We count both types of separating edges in each 4-tuple of two blue and two red points. The results are listed in Table 1.

Let c_{M} be the number of monochromatic separating edges and c_{B} the number of bicolored separating edges, summing over all 4-tuples of P. Furthermore, denote the number of 4-tuples in the configuration seen in Figure 11 by T_{A}, the number of 4-tuples in the configuration seen in Figure 12 by T_{B}, and the number of 4-tuples in the configuration seen in Figure 13 and 14 by T_{C}. Then, $T_{\mathrm{B}} + T_{\mathrm{C}} = c_{\mathrm{M}}$ and $T_{\mathrm{A}} + 2T_{\mathrm{C}} = c_{\mathrm{B}}$. Counting all 4-tuples gives

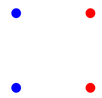
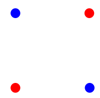
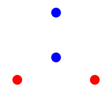
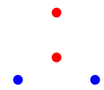
	Bicolored order type			
	convex position		non-convex position	
separating edges	 **Figure 11** T_{A}	 **Figure 12** T_{B}	 **Figure 13** T_{C}	 **Figure 14** T_{C}
monochromatic	0	2	1	1
bicolored	2	0	2	2

Table 1 The number of separating edges broken down to order type and colors. Figure 11: All points lie in convex position, the color classes are linearly separable. Figure 12: All points lie in convex position and are colored alternately. Figure 13, 14: The points are in non-convex position, all possible colorings are equivalent.

$T_{\mathrm{A}} + T_{\mathrm{B}} + T_{\mathrm{C}} = \binom{m}{2}\binom{n}{2}$. Since the configuration depicted in Figure 11 is the only one that admits a crossing, the number T_{A} of these 4-tuples gives the rectilinear crossing number. Therefore, we solve the system of linear equations to get T_{A} and obtain

$$\overline{\mathrm{cr}}(P) = T_{\mathrm{A}} = 2\binom{m}{2}\binom{n}{2} - c_{\mathrm{M}} - \frac{c_{\mathrm{B}}}{2}. \tag{3}$$

So as soon as we know c_{B} and c_{M}, we can compute the rectilinear crossing number of P. Using λ_B and λ_R at the indices corresponding to points of the same color we sum over all monochromatic separating edges and get

$$c_{\mathrm{M}} = \sum_{\substack{b_i,b_j\\ i<j}} \lambda_R(b_i,b_j)\,\lambda_R(b_j,b_i) + \sum_{\substack{r_k,r_l\\ k<l}} \lambda_B(r_k,r_l)\,\lambda_B(r_l,r_k).$$

We count the bicolored separating edges in all 4-tuples in a similar manner:

$$c_{\mathrm{B}} = \sum_{b_i,r_k} \lambda_B(b_i,r_k)\,\lambda_R(r_k,b_i) + \lambda_R(b_i,r_k)\,\lambda_B(r_k,b_i).$$

To compute c_{M} and c_{B}, all $\binom{m}{2} + \binom{n}{2}$ monochromatic point tuples, respectively all mn bicolored point tuples, are considered separately. Therefore, given λ_B and λ_R, the two constants can be computed in $\Theta\left(m^2+n^2\right)$ time. This is significantly faster than the brute-force approach of testing all pairs of bicolored edges for crossings, which takes $\Theta\left(\binom{m}{2}\binom{n}{2}\right) = \Theta\left(m^2n^2\right)$ time. Inserting c_{M} and c_{B} into Equation (3) gives the (rectilinear) crossing number of the complete bipartite graph on P.

▶ **Theorem 11.** *Given a bicolored point set $P = B \,\dot\cup\, R$ and λ_B and λ_R, the rectilinear crossing number of the bipartite straight-line graph drawn on P is given by*

$$\overline{\mathrm{cr}}(P) = 2\binom{m}{2}\binom{n}{2} - c_M - \frac{c_B}{2}, \tag{4}$$

and can be computed in $\Theta\left(m^2+n^2\right)$ time.

Let us elaborate this with two examples given as the two bicolored order types in Figure 15. For each of them, we count the separating edges of all types. The point set in Figure 15(a)

(a) 4 monochromatic separating edges (1 red, 3 blue), 10 bicolored separating edges

(b) 5 monochromatic separating edges (2 red, 3 blue), 10 bicolored separating edges

Figure 15 Examples for computing the rectilinear crossing number by using separating edges.

has 4 monochromatic separating edges (1 red, 3 blue) and 10 bicolored separating edges. Using Equation (4), its rectilinear crossing number is computed as follows:

$$\overline{\mathrm{cr}}(P) = 2\binom{2}{2}\binom{4}{2} - 4 - \frac{10}{2} = 2 \cdot 1 \cdot 6 - 4 - 5 = 3.$$

For the bicolored order type in Figure 15(b) we count 5 monochromatic separating edges (2 red, 3 blue) and 10 bicolored separating edges, and get

$$\overline{\mathrm{cr}}(P) = 2\binom{2}{2}\binom{4}{2} - 5 - \frac{10}{2} = 2 \cdot 1 \cdot 6 - 5 - 5 = 2.$$

5 All Bicolored Order Types of Small Cardinality

To obtain all bicolored order types of small cardinality we use a straight-forward approach. We take all uncolored order types on $|S| = m + n$ points as provided by [2] and color m vertices blue and n vertices red in all possible ways. This results in many duplicates—because of symmetries and because the same bicolored order type might be obtained from different uncolored order types—which we filter out. The resulting numbers of bicolored order types for up to $m+n = 10$ points are listed in Table 2. Note that here we consider unlabelled point sets. Thus, two sets belong to the same order type if there are labellings of the sets such that all triple orientations are the same, or all triple orientations are inverse. Point sets where all (bicolored) triple orientations are inverse have the same combinatorial characteristics, for example the convex hull and the intersection of edges. Since we want to generate as few sets as possible, we do not distinguish between such sets.

On the one hand we do not consider several of the triple orientations of an (uncolored) order type as these points belong to the same color class. Thus, previously different order types might belong to the same equivalence class for colored order types. On the other hand, adding color to the points might generate different colored order types out of one (uncolored) order type. Therefore, it is ad hoc not clear whether for some fixed cardinality and for a given relation $m : n$ of blue and red points there are more colored or uncolored order types. The results in the table suggest that only for a very unbalanced color distribution there are less colored order types. The number of colored order types with only one red point is equivalent to the number of self-dual 2-colored necklaces with $2(m + n)$ beads [5] and also to the number of combinatorial types of simplicial neighborly polytopes in dimension $2(m + n) - 3$ with $2(m + n)$ vertices, see Sequence A007147 of The On-Line Encyclopedia of Integer Sequences [14]. Already for 2 red points in a bicolored order type we are not aware of a way to determine their number for some fixed cardinality $m + 2$ without enumerating them.

$\lvert S \rvert$	m	n	# bicolored order types	# order types
4	2	2	3	
	3	1	2	
			Σ: 5	2
5	3	2	12	
	4	1	2	
			Σ: 14	3
6	3	3	72	
	4	2	68	
	5	1	4	
			Σ: 144	16
7	4	3	2 108	
	5	2	422	
	6	1	5	
			Σ: 2 535	135
8	4	4	46 715	
	5	3	44 397	
	6	2	3 495	
	7	1	9	
			Σ: 94 616	3 315
9	5	4	5 088 553	
	6	3	1 148 398	
	7	2	33 193	
	8	1	12	
			Σ: 6 270 156	158 817
10	5	5	343 385 532	
	6	4	342 917 794	
	7	3	35 582 251	
	8	2	362 625	
	9	1	23	
			Σ: 722 248 225	14 309 547

Table 2 Number of bicolored order types of size (m, n), $m \geq n$, $m + n \leq 10$, compared to the number of uncolored order types on $m + n$ points. Here unlabelled point sets are considered and also sets where all triple orientations are inverse belong to the same order type.

Since the time needed to compute all colorings and to filter out duplicates increases tremendously as the order type gets larger, this approach to generate colored order types is not further pursued. In future work, we plan to develop more sophisticated methods for enumerating all bicolored order types that are of interest for specific applications like, for example, minimizing the number of crossings.

6 Future Work

Although the algorithms in this paper are not optimized, they all take polynomial time and serve as a proof of concept. Moreover, all our approaches only use the abstract information of the colored point triples, and no geometric information of the point sets. Thus, in future work we plan to focus on the performance of algorithms using bicolored order types, with respect to running time and in which form the information of the bicolored order type is stored. To this end, we will also consider other axiom systems that define uniform acyclic rank-3 oriented matroids—as mentioned in the introduction—or dual structures (e.g. colored wiring diagrams), and see how some specific information, like the number of bicolored crossings, can be read directly from the wiring diagram. Extending only those wiring diagrams that are dual to a point set with, for example, a small crossing number, will lead to more efficient algorithms to generate bicolored order types that are of interest for specific applications.

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