Worst-Case Optimal Covering of Triangles by Disks

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Abstract

The *critical covering area* of a triangle $\mathcal{T}$ is the value $A^*(\mathcal{T})$ such that (1) each set of disks with a total area of at least $A^*(\mathcal{T})$ permits a covering of $\mathcal{T}$ and (2) for each $A^- < A^*(\mathcal{T})$, there is a set D^- of disks with a total area of A^- such that D^- cannot cover $\mathcal{T}$. The *critical covering coefficient* $C^*(\mathcal{T})$ of $\mathcal{T}$ is the ratio $A^*(\mathcal{T})/\|\mathcal{T}\|$ between the critical covering area of $\mathcal{T}$ and the area of $\mathcal{T}$.

In this work, we prove that the critical covering coefficient for isosceles triangles with an apex angle $\alpha \leq \pi/4$ is $\frac{\pi \tan(\alpha/2)}{\sin^2(\alpha)}$, and the covering coefficient for isosceles triangles with an apex angle $\alpha \geq \pi/2$ is $\pi \tan(\alpha/2)$. The critical covering coefficient for equilateral triangles is $\pi\sqrt{3}/2 \approx 2.7207\ldots$; the comparison with the recently established critical covering coefficient for squares ($\frac{195\pi}{256} \approx 2.39301\ldots$, as shown by Fekete et al. in 2020) indicates the additional difficulty of covering triangles. As a corollary, we obtain that $\frac{\pi}{h}$ is the critical covering area of obtuse triangles with inner angles $\alpha_1 \geq \frac{\pi}{2}$ and $\frac{\pi}{3} \geq \alpha_2, \alpha_3$ and a height of h passing the corner with an inner angle of α_1.

Our proofs are constructive, i.e., we provide corresponding worst-case optimal covering algorithms.

Keywords and phrases Geometric optimization, worst-case optimal, disk covering, triangles

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Supplementary Material `https://github.com/phillip-keldenich/circlecover-triangles`

1 Introduction

Given a set of (not necessarily equal) disks, is it possible to arrange them so that they completely cover a given region, such as a triangle? Covering problems of this type are of fundamental theoretical interest, but also have a variety of different applications, most notably in sensor networks, communication networks, wireless communication, surveillance, robotics, and even gardening and sports facility management.

If the total area of the disks is too small, it is clear that completely covering the region is impossible. On the other hand, if the total disk area is sufficiently large, finding a covering seems intuitively easy; however, for thin triangles, a major fraction of the covering disks may be useless, so a relatively large total disk area may be required. The same issue is of obvious importance for applications: What fraction of the total cost of disks can be put to efficient use for covering? This motivates the question of characterizing a critical threshold: Find the minimum value for which any set of disks with total area at least A can cover an equilateral triangle or a halfsquare, i.e., an isosceles, right-angled triangle? In this paper we provide worst-case optimal algorithms for both scenarios.

1.1 Related work

Like many other packing and covering problems, disk covering is typically quite difficult, compounded by the geometric complications of dealing with irrational coordinates that arise when arranging circular objects. This is reflected by the limitations of provably optimal results for the largest disk, square or triangle that can be covered by n unit disks, and hence, the “thinnest” disk covering, i.e., a covering of optimal density. As early as 1915, Neville [35] computed the optimal arrangement for covering a disk by five unit disks, but reported a wrong optimal value; much later, Bezdek[6, 7] gave the correct value for $n = 5, 6$. As recently as 2005, Fejes Tóth [43] established optimal values for $n = 8, 9, 10$. The question of incomplete coverings was raised in 2008 by Connelly who asked how one should place n small disks of radius r to cover the largest possible area of a disk of radius $R > r$. Szalkai [42] gave an optimal solution for $n = 3$. For covering rectangles by n unit disks, Heppes and Mellissen [26] gave optimal solutions for $n \leq 5$; Melissen and Schuur [32] extended this for $n = 6, 7$. See the website by Friedman [23] for illustrations of the best known solutions for $n \leq 12$. Along similar lines, covering equilateral triangles by n unit disks has also been studied. Melissen [31] gave optimality results for $n \leq 10$, and conjectures for $n \leq 18$; the difficulty of these seemingly small problems is illustrated by the fact that Nurmela [36] gave conjectured optimal solutions for $n \leq 36$, improving $n = 13$ from Melissen. Carmi et al. [10] considered algorithms for covering point sets by unit disks at fixed locations. There are numerous other related problems and results; for relevant surveys, see Fejes Tóth [14] (Section 8), Fejes Tóth [44] (Chapter 2), Brass et al. [9] (Chapter 2) and the book by Böröczky [8].

Even less is known for covering by non-uniform disks, with the majority of previous research focusing on algorithmic aspects. Alt et al. [3] gave algorithmic results for minimum-cost covering of point sets by disks, where the cost function is $\sum_j r_j^\alpha$ for some $\alpha > 1$, which includes the case of total disk area for $\alpha = 2$. Agnetis et al. [2] discussed covering a line segment with variable radius disks. Along similar lines, Abu-Affash et al. [1] studied covering a polygon minimizing the sum of areas, and Bánhelyi et al. [4] gave algorithmic results for the covering of polygons by variable disks with prescribed centers.

For relevant applications, we mention the survey by Huang and Tseng [27] for wireless sensor networks, the work by Johnson et al. [28] on covering density for sensor networks, the algorithmic results for placing a given number of base stations to cover a square [11] and a convex region by Das et al. [12]. For minimum-cost sensor coverage of planar regions, see Xu et al. [45]; for wireless communication coverage of a square, see Singh and Sengupta [40], and Palatinus and Bánhelyi [38] for the context of telecommunication networks.

The analogous question of *packing* a set of unit disks into a square has also attracted considerable attention. For the case of $k = 13$, the optimal value for the densest square covering was only established in 2003 [22], while the optimal value for 14 unit disks is still unproven; densest packings of n disks in equilateral triangles are subject to a long-standing conjecture by Erdős and Oler from 1961 [37] that is still open for $n = 15$. For other examples of mathematical work on densely packing relatively small numbers of identical disks, see [24, 30, 20, 21], and [39, 29, 25] for related experimental work. The best known solutions for packing equal disks into squares, triangles and other shapes are continuously published on Specht’s website `http://packomania.com` [41].

The critical density for packing (not necessarily equal) squares into a unit square was introduced in 1967 by Moon and Moser [33], who used a shelf-packing approach to establish the value of $1/2$. Establishing the critical packing density for (not necessarily equal) disks in a square was proposed by Demaine, Fekete, and Lang [13] and solved by Morr, Fekete and Scheffer [34, 19]. Using a recursive procedure for cutting the container into triangular pieces, they proved that the critical packing density of disks in a square is $\frac{\pi}{3+2\sqrt{2}} \approx 0.539$; see the video [5] for an animated overview. They also established the critical packing density for circles in a non-acute triangle with side lengths x, y, and y as

$$\phi_t = \pi\sqrt{\frac{(x+y-z)(z+x-y)(y+z-x)}{(x+y+z)^3}}. \tag{1}$$

More recently, Fekete, Keldenich and Scheffer [17] established the critical packing densities for packing disks into disks as $1/2$. For packing squares into disks, Fekete et al. [16] proved that the critical density for packing squares into a disk is $8/5\pi \approx 0.509\ldots$.

Closely related to this paper is the work by Fekete et al. [15] that provides a full characterization of the critical area $A^*(\lambda)$ that is sometimes necessary and always sufficient for covering a $\lambda \times 1$ rectangle with $\lambda \geq 1$: There is a threshold value $\lambda_2 = \sqrt{\sqrt{7}/2 - 1/4} \approx 1.035797\ldots$, such that for $\lambda < \lambda_2$ the critical covering area $A^*(\lambda)$ is $A^*(\lambda) = 3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right)$, and for $\lambda \geq \lambda_2$, the critical area is $A^*(\lambda) = \pi(\lambda^2+2)/4$; these values are tight. For the special case $\lambda = 1$, i.e., for covering a unit square, the critical covering area is $\frac{195\pi}{256} \approx 2.39301\ldots$. See Theorem 4 and Lemma 5, and the video [18] for a detailed visualization.

1.2 Our results

We are given a triangular container $\mathcal{T}$. For a set $D = \{r_1, \ldots, r_n\}$ of radii $r_1 \geq r_2 \geq \cdots \geq r_n$, we want to decide whether there is a placement of n disks with radii $r_1, \ldots, r_n$ on $\mathcal{T}$, such that every point $x \in \mathcal{T}$ is covered by at least one disk.

The *critical covering area* of a shape $\mathcal{S}$ is the value $A^*(\mathcal{S})$ such that (1) each set of disks with total area of at least $A^*(\mathcal{S})$ permits a covering of $\mathcal{S}$ and (2) for each $A' < A^*(\mathcal{S})$, there is a set D^- of disks with total area A' such that D^- cannot cover $\mathcal{T}$. The *critical covering coefficient* $C^*(\mathcal{S})$ of $\mathcal{S}$ is the ratio $A^*(\mathcal{S})/\|\mathcal{S}\|$ of the critical covering area of $\mathcal{S}$ and the area of $\mathcal{S}$.

▶ **Theorem 1.** *The critical covering coefficient of isosceles triangles with apex angle* $0 < \alpha \leq \pi/4$ *is* $\frac{\pi\tan(\alpha/2)}{\sin^2(\alpha)}$.

▶ **Theorem 2.** *The critical covering coefficient of equilateral triangles is* $\frac{2\pi}{\sqrt{3}}$.

▶ **Theorem 3.** *For* $\alpha \geq \pi/2$, *the critical covering coefficient for an isosceles triangle with apex angle* α *is* $\pi\tan(\alpha/2)$.

2 Preliminaries

For any set D of disks, the *total disk area* is $A(D) := \pi\sum_{r\in D} r^2$. The *weight* of a disk of radius r is r^2, and $w(D) := \frac{A(D)}{\pi}$ is the *total weight* of D.

For proving our results, we use Greedy Splitting for partitioning a set of disks into two sets whose weight differs by at most the weight of the smallest disk in the heavier set: After sorting the disks by decreasing radius, we start with two empty sets and continue to place the next disk in the set with smaller total weight.

2.1 Critical covering area for rectangles

In a recent paper [15], we give a complete characterization of the critical covering area for arbitrary rectangles, as follows.

▶ **Theorem 4** ([15], Theorem 1). *Let $\lambda \geq 1$ and $\mathcal{R}$ a $1 \times \lambda$-rectangle. For $\lambda_2 = \sqrt{\frac{\sqrt{7}}{2} - \frac{1}{4}} \approx 1.035797\ldots$, the critical covering area of $\mathcal{R}$ is*

$$A^*(\lambda) = \begin{cases} 3\pi\left(\frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256\lambda^2}\right) & \text{if } \lambda < \lambda_2, \\ \pi\frac{\lambda^2+2}{4} & \text{otherwise.} \end{cases}$$

The proof of Theorem 4 is constructive, i.e., it yields an efficient worst-case optimal algorithm for covering arbitrary $1 \times \lambda$-rectangles. This algorithm is worst-case optimal for every choice of λ.

Furthermore, in [15], we prove the following lemmas which provide improved, i.e., decreased critical covering area under the assumption that each disks' weight upper-bounded by some threshold.

▶ **Lemma 5** ([15], Lemma 3). *Let $\hat{\sigma} \coloneqq \frac{195\sqrt{5257}}{16384} \approx 0.8629$. Let $\sigma \geq \hat{\sigma}$ and $E(\sigma) \coloneqq \frac{1}{2}\sqrt{\sqrt{\sigma^2+1}+1}$. Let $\lambda \geq 1$ and $D = \{r_1, \ldots, r_n\}$ be any collection of disks with $\sigma \geq r_1^2 \geq \ldots \geq r_n^2$ and $w(D) = \sum_{i=1}^{n} r_i^2 \geq E(\sigma)\lambda$. Then D can cover a rectangle $\mathcal{R}$ of dimensions $\lambda \times 1$.*

▶ **Lemma 6** ([15], Lemma 4). *Let $\mathcal{R}$ be an $\ell \times s$-rectangle with $\ell \geq s$. Let D be a set of disks such that the largest disk satisfies $r_1 \leq 0.375s$. Then D can cover $\mathcal{R}$ if its total area $A(D)$ is at least $0.61\pi\ell s$.*

2.2 Computer-assisted proofs via interval arithmetic

For some special cases of our proofs, we make use of computer assistance by applying interval arithmetic to prove certain inequalities. Generally, to prove an inequality $c \leq f(x_1, \ldots, x_k) \leq d$ for some fixed function $f : \mathbb{R}^k \to \mathbb{R}$, constants $c, d \in \mathbb{R}$ and variables $x_1, \ldots, x_k$ for all values $(x_1, \ldots, x_k) \in [a_1, b_1] \times \cdots \times [a_k, b_k]$ for some bounds $a_1, b_1, \ldots, a_k, b_k \in \mathbb{R}$, we subdivide the input space $[a_1, b_1] \times \cdots \times [a_k, b_k]$ into sufficiently many pieces which together cover the entire input space. This can be done, for instance, by subdividing each of the input intervals $[a_i, b_i]$ into some number k_i of pieces of equal width, resulting in $k_1 \cdots k_k$ pieces in total. For each piece $P = [p_1, q_1] \times \cdots \times [p_k, q_k]$, we then evaluate

$$f([p_1, q_1], \ldots, [p_k, q_k]) \coloneqq \left[\inf_{x_i \in [p_i, q_i]} f(x_1, \ldots, x_k), \sup_{x_i \in [p_i, q_i]} f(x_1, \ldots, x_k)\right].$$

and check whether the resulting interval is contained in $[c, d]$. If this succeeds for all pieces, then clearly the inequality must hold for all input values. For monotonic functions, evaluating them on an interval can be done by evaluating the function at the boundaries. Note that we may use any interval $I \supseteq f([p_1, q_1], \ldots, [p_k, q_k])$ instead of the exact bounds on f, as long as $I \subseteq [c, d]$. For instance, to evaluate a function $f(x) = g(x) - h(x)$ on $x \in [a, b]$, we may choose to compute $[g_a, g_b] = g([a, b])$ and $[h_a, h_b] = h([a, b])$ exactly, but use $I = [g_a - h_b, g_b - h_a]$ instead of the (possibly proper) subset $f([a, b])$. This enables us to compute lower and upper bounds, even for intricate non-monotonic real functions, which are sufficiently tight to prove certain inequalities of interest. It also allows us to handle numerical issues in order to maintain soundness in the presence of errors resulting from the use of limited precision floating-point numbers.

For some lower-level details regarding our particular implementation of interval arithmetic, see Section 4.4.

3 High-level overview

We independently prove each of the main theorems (Theorems 1–3) using two bounds. On one hand, for each theorem, a worst-case instance shows that the critical covering coefficient cannot be lower than the claimed value. On the other hand, for each theorem, we develop a recursive algorithm to provide the matching upper bound as follows. We prove that, given any set of disks with at least the claimed critical covering area, the algorithm is *guaranteed* to compute a feasible covering of the triangle. This then proves that the claimed worst-case instance is actually the worst case. In each case, we prove this guarantee by induction on the number of disks, following the recursive structure of the algorithm.

In each proof, w.l.o.g., we assume the set of disks $D = \{r_1, \dots, r_n\}$ to have a total area that is exactly the claimed critical covering area; otherwise, we have nothing to prove or could simply reduce the size of the input disks appropriately. Furthermore, we assume $r_1 \geq r_2 \geq \cdots \geq r_n$. The induction step generally consists of a case distinction, each case corresponding to a different approach to covering the triangle with the given disks. For instance, placing the largest disk r_1 such that it covers the entire base of the triangle is often a good approach, but it clearly only works for sufficiently large r_1.

For Theorems 2 and 3, this case distinction involves only a managable number of cases depending on r_1 and sometimes r_2; these proofs are carried out without computer aid. For Theorem 1, the case distinction is more involved and depends on the apex angle α and the three largest disks r_1, r_2 and r_3. This results in a relatively large number of cases, for which the analysis is carried out using computer assistance, using interval arithmetic as outlined in Section 2.2.[1] Note that our results are *tight* in the numerical sense, i.e., there is no gap between the lower and the upper bound. Therefore, at any fixed precision, a method that introduces discretization and rounding errors such as interval arithmetic cannot completely cover all cases on its own, as it needs some amount of excess disk area to account for these errors. For instance, cases arbitrarily close to the worst case instances cannot simply be handled in this manner, as well as situations with apex angle α arbitrarily close to 0; we outline how we avoid these situations in Section 3.1.

In the following, we give a high-level overview of the worst-case instances and the recursive algorithms for each of the proofs. The details are presented in Sections 4.1–4.3.

3.1 Covering isoceles triangles with small apex angle

In this section, we sketch the proof of Theorem 1.

▶ **Theorem 1.** *The critical covering coefficient of isosceles triangles with apex angle* $0 < \alpha \leq {}^{\pi}/_{4}$ *is* $\frac{\pi \tan({}^{\alpha}/_{2})}{\sin^2(\alpha)}$.

W.l.o.g., we assume that base of the triangle has length 1 and is vertical, as depicted in Figure 1(a); all other cases can be handled by rotating and scaling the triangle and disks. For

[1] The code for all interval arithmetic-based proofs can be found at `https://github.com/phillip-keldenich/circlecover-triangles`.

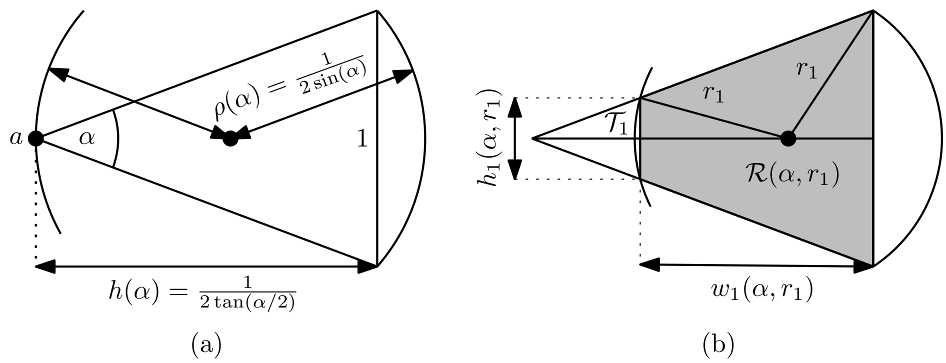

Figure 1 (a) An isosceles triangle with apex a, apex angle α, base length 1 and height $h(\alpha)$. The radius of the smallest disk completely covering the triangle is $\rho(\alpha) = 1/2\sin(\alpha)$. (b) Placing r_1 such that it covers a trapezoid $\mathcal{R}(\alpha, r_1)$ of width $w_1(\alpha, r_1)$, right height 1 and left height $h_1(\alpha, r_1)$. This placement only works for $r_1 \geq 1/2$; Lemma 7 shows that we can successfully place r_1 in this manner for all $r_1 \geq 1/2$ that we need to consider.

an isosceles triangle with apex angle α, let $\rho(\alpha)$ denote the radius of a single disk that barely suffices to cover the triangle. Our worst-case instance consists of a single disk $r_1 = \rho(\alpha)$; making r_1 any smaller would result in an instance in which the triangle cannot be covered. The area of r_1 is $\frac{\pi}{4\sin^2(\alpha)}$ for a triangle area of $\frac{1}{4\tan(\alpha/2)}$, yielding a lower bound of $\frac{\pi\tan(\alpha/2)}{\sin^2(\alpha)}$ on the critical covering coefficient.

The recursive covering algorithm that provides the matching upper bound, and thus proves that this is indeed the worst possible case, consists of several *covering routines*. Each covering routine fixes the placement of some number of disks and then possibly recurses on remaining, scaled down versions of the input triangle that cover the entire input region together with the placed disks. Each covering routine has an associated *success criterion* that is a sufficient condition for the routine to be successful. This success criterion depends only on α and the three largest radii r_1, r_2, r_3. Our algorithm simply iterates through the list of covering routines and applies the first one for which the success criterion is met. We use a combination of manual analysis and computer-assisted interval arithmetic to prove that for any possible combination of values α, r_1, r_2, r_3, there is always at least one success criterion that is met. Together with the induction hypothesis, this guarantees the overall success of the algorithm.

In the following, we sketch the covering routines and their success criteria. The first, most straightforward covering routine is called Base Trapezoid and places the largest disk r_1 on the leftmost point on the vertical center line of the triangle such that it covers a trapezoid $\mathcal{R}(\alpha, r_1)$ including the entire base of the triangle; see Figure 1(b). It then recursively applies the algorithm to cover the remaining triangle $\mathcal{T}_1$ with the remaining disks $r_2, \ldots$, if any triangle and any disks remain.

▶ **Lemma 7.** *For $\alpha < \pi/2$ and any $1/2 \leq r_1 \leq \rho(\alpha)$, the area $\|\mathcal{R}(\alpha, r_1)\|$ satisfies $\frac{\pi r_1^2}{\|\mathcal{R}(\alpha,r_1)\|} \leq \frac{\pi\tan(\alpha/2)}{\sin^2(\alpha)}$, i.e., covering $\mathcal{R}(\alpha, r_1)$ with r_1 is at least as efficient as covering the entire triangle with a disk of radius $\rho(\alpha)$.*

Proof. The distance between the right and the left side of the trapezoid $\mathcal{R}(\alpha, r_1)$ is

$$w(\alpha, r_1) = \frac{\sqrt{4r_1^2 - 1}\,(\cos(\alpha) + 1)}{2} + \frac{\sin(\alpha)}{2}.$$

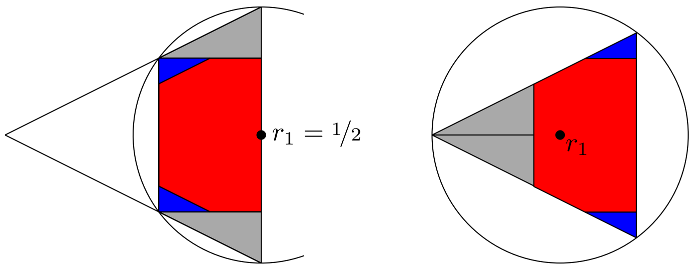

Figure 2 The trapezoid covered by a disk of radius 1/2 has the same area as the maximum isosceles triangle with apex angle α that can be covered by a disk of radius 1/2.

The height of $\mathcal{R}(\alpha, r_1)$ at the left side is

$$h_\ell(\alpha, r_1) = 1 - \left(\sqrt{4r_1^2 - 1}\cos(\alpha) + \sqrt{4r_1^2 - 1} + \sin(\alpha)\right)\tan\left(\frac{\alpha}{2}\right).$$

Therefore, the area $E(\alpha, r_1) = \frac{\mathcal{R}(\alpha, r_1)}{r_1^2}$ of $\mathcal{R}(\alpha, r_1)$ covered per unit of weight of r_1 is

$$\frac{\left(\sqrt{4r_1^2 - 1}\,(\cos(\alpha) + 1) + \sin(\alpha)\right)\left(2 - \left(\sqrt{4r_1^2 - 1}\cos(\alpha) + \sqrt{4r_1^2 - 1} + \sin(\alpha)\right)\tan\left(\frac{\alpha}{2}\right)\right)}{4r_1^2}.$$

In the following, we call $E(\alpha, r_1)$ the *efficiency* of covering $\mathcal{R}(\alpha, r_1)$ by r_1. We show $E(\alpha, r_1) \geq \frac{\sin^2(\alpha)}{\tan(\alpha/2)}$, implying our claim. The real solutions of $E(\alpha, r_1) - \frac{\sin^2(\alpha)}{\tan(\alpha/2)} = 0$ for r_1 are

$$-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2\sqrt{\sin^2(\alpha)}}, \frac{1}{2\sqrt{\sin^2(\alpha)}}.$$

Two of these are negative and can be discarded; furthermore, $\sin(\alpha)$ is positive for all α we have to consider, thus the only two remaining solutions can be written as $r_1 = \frac{1}{2}$ and $r_1 = \frac{1}{2\sin(\alpha)} = \rho(\alpha)$. This means that a covering of the trapezoid $\mathcal{R}(\alpha, 1/2)$ and a covering of an isosceles triangle with apex angle α by a single disk have the same efficiency; see Figure 2.

Intuitively, our claim follows from this by the observation that any radius in between these two extremes covers $\mathcal{R}(\alpha, r_1)$ more efficiently. More precisely, for some r_1 sufficiently closely below $\rho(\alpha)$, $E(\alpha, r_1) - \frac{\sin^2(\alpha)}{\tan(\alpha/2)}$ must be positive, because the weight of r_1 continues to grow quadratically with r_1, but the additional area covered by increasing r_1 tends to 0 for $r_1 \to \rho(\alpha)$. Thus, by the intermediate value theorem, $E(\alpha, r_1) - \frac{\sin^2(\alpha)}{\tan(\alpha/2)}$ is positive between its roots $r_1 = \frac{1}{2}$ and $r_1 = \rho(\alpha)$; otherwise, there would have to be another root in between. ◀

A consequence of Lemma 7 is that $r_1 \geq 1/2$ is a sufficient success criterion for Base Trapezoid: the amount of disk area remaining after placing r_1 is always at least as much as an inductive application of Theorem 1 requires. This allows us to remove any case of $r_1 \geq 1/2$ from further consideration. Being able to bound $r_1 < 1/2$ removes all cases that are close to the worst case from consideration, which is, as outlined above, particularly important for the computer-assisted part of the proof.

This allows us to focus on the case $r_1 < 1/2$ for the remaining covering routines. The second covering routine is called Rectangle Cover and makes use of Lemma 5; essentially, we compute the bounding box of our triangle and apply our rectangle covering algorithm.

Because the shorter side of the bounding box has length 1 and we have $r_1 < 1/2$, we can successfully cover the rectangle by Lemma 5 if the total disk area is large enough, i.e., if $\frac{\pi}{4\sin^2(\alpha)} \geq \frac{195\pi}{256} h(\alpha) = \frac{195\pi}{512\tan(\alpha/2)}$. Solving for α yields a threshold below which the efficiency guaranteed by Lemma 5 is sufficient to allow us to cover the bounding box of our triangle instead of the triangle itself. To improve readability, we do not present the extensive exact representation of that threshold; instead, we approximate it by the largest IEEE 754 double value

$$\alpha_r = 0.3449678733707022271204323260462842881679534912109375$$

below the actual threshold, which is accurate to 16 digits.

This allows us to restrict our attention to $\alpha \in [\alpha_r, \pi/4]$, thus removing another problematic case, namely $\alpha \to 0$, from consideration. For α in that range and $r_1 < 1/2$, we use three covering routines called Rectangle Base Cover, R1 Center and Two Large Disks, see Figures 3 and 4. Rectangle Base Cover uses Lemma 6 to cover a rectangle $\mathcal{R}_b$

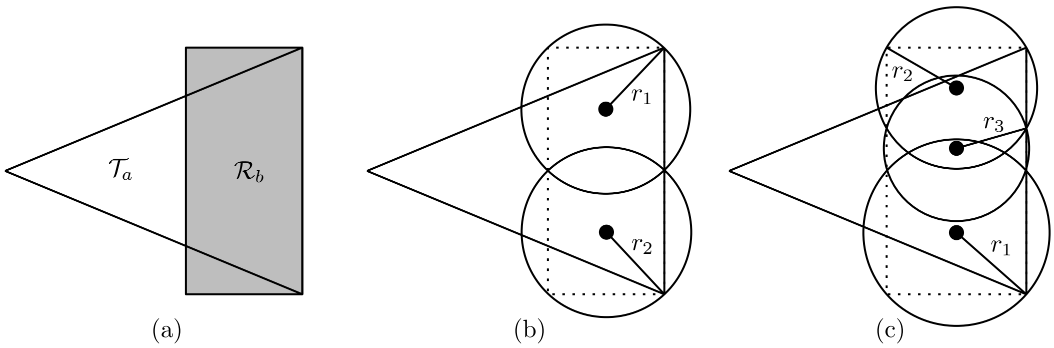

Figure 3 Rectangle Base Cover either covers a rectangle at the base of the triangle using Lemma 6 (a) or r_1, r_2 and r_3 (b,c).

at the base of the triangle with a subset of the disks as depicted in Figure 3(a). As an alternative to Lemma 6, we also consider covering $\mathcal{R}_b$ using r_1, r_2 and potentially r_3; see Figure 3(b) and (c). In any case, disks that are not used to cover $\mathcal{R}_b$ are used to recursively cover the remaining triangle. The success criterion for Rectangle Base Cover works as follows.

Covering $\mathcal{R}_b$ using Lemma 6 We begin by picking a start disk $r_i \in \{r_1, r_2, r_3\}$; we consider each choice. We then compute the goal width $w_B = r_i/0.375$ of $\mathcal{R}_b$ and height $h_B = \max\{1, w_B\}$ of $\mathcal{R}_b$. Starting with some disk r_i, we iteratively collect disks D_b in non-increasing order of radius until their total weight $w(D_b)$ exceeds $0.61 w_B h_B$. We then apply Lemma 6 to cover a rectangle $\mathcal{R}'_b \supseteq \mathcal{R}_b$ of height h_B and width $w'_B = \frac{w(D_b)}{0.61 h_B}$. If we run out of disks in the process, the covering routine fails for the current choice of r_i.

We then compute the efficiency of the covering w.r.t. the triangle area covered by $\mathcal{R}'_b$; if we cover at least $\frac{\sin^2(\alpha)}{\tan(\alpha/2)}$ triangle area per unit of weight, we can successfully recurse on $\mathcal{T}_a = \mathcal{T} \setminus \mathcal{R}'_b$; otherwise, the covering routine fails for the current choice of r_i.

Similarly to our algorithm, our success criterion for this routine considers each choice $r_i \in \{r_1, r_2, r_3\}$ and computes w_B and h_B for it. We then check whether there is enough weight in the remaining disks such that we cannot run out of disks before exceeding weight $0.61 w_B h_B$. Furthermore, we bound the actual weight of D_b by $w(D_b) \leq 0.61 w_B h_B + r_i^2$. This allows us to compute an upper bound on the actual width w'_B and thus a lower bound on the

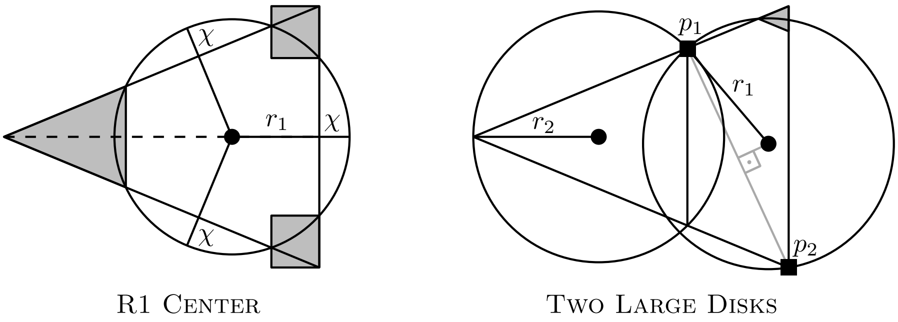

Figure 4 The covering routine R1 Center places r_1 on the vertical center line of the triangle such that r_1 intersects all three sides and the distance χ that the disks protrudes from the triangle is the same on all three sides. The covering routine Two Large Disks uses r_2 to cover an isosceles triangle containing the apex and uses r_1 to cover the line segment p_1p_2 between the top-right corner of the triangle covered by r_2 and the bottom-right corner of the input triangle, shifting r_1 as far away from that line segment as possible.

efficiency, i.e., the disk weight used per triangle area, because the efficiency is monotonically decreasing in w'_B. If this lower bound is at least $\frac{\sin^2(\alpha)}{\tan(\alpha/2)}$, our success criterion is satisfied.

Covering $\mathcal{R}_b$ using r_1 and r_2 The maximum rectangle of height 1 that can be covered by two disks has width

$$w_{\max}(r_1, r_2) = \frac{\sqrt{-16r_1^4 + 32r_1^2r_2^2 + 8r_1^2 - 16r_2^4 + 8r_2^2 - 1}}{2}.$$

We can thus check whether it is possible to cover $\mathcal{R}_b$ with two disks, and compute the efficiency of such a covering. If the efficiency is at least $\frac{\sin^2(\alpha)}{\tan(\alpha/2)}$ triangle area per unit of weight, we can successfully recurse on $\mathcal{T}_a$.

Covering $\mathcal{R}_b$ using r_1, r_2 and r_3 In this case, we compute the total remaining area of disks $D \setminus \{r_1, r_2, r_3\}$, and compute the height h' of the largest isosceles triangle with apex angle α for which Theorem 1 guarantees successful recursion. This allows us to compute the width $w_B = h(\alpha) - h'$ of the remaining rectangle. We can then check whether $\sqrt{4r_1^2 - w_B^2} + \sqrt{4r_2^2 - w_B^2} + \sqrt{4r_3^2 - w_B^2} \geq 1$, i.e., whether the three disks r_1, r_2, r_3 can cover $\mathcal{R}_b$.

For details on the remaining two covering routines R1 Center and Two Large Disks, refer to Section 4.1.

3.2 Covering equilateral triangles

In this section, we sketch the proof of Theorem 2; for details, refer to Section 4.2.

▶ **Theorem 2.** *The critical covering coefficient of equilateral triangles is* $\frac{2\pi}{\sqrt{3}}$.

We denote the corners of our equilateral triangle $\mathcal{T}$ by a, b, and c. W.l.o.g., we assume the sides of $\mathcal{T}$ to have length 1; thus, $\mathcal{T}$ has area $\sqrt{3}/4$. Furthermore, we assume that bc lies horizontal, and that a lies above bc, see Figure 5(a). All other cases can be handled by scaling and rotating.

The worst case instance D_{worst} consists of two disks with radius $1/2$, see Figure 5(a). If we shrink the two disks by any $\varepsilon > 0$, each of them can cover at most one corner of $\mathcal{T}$, leaving at least one corner uncovered. The total area of D_{worst} is $\pi/2$, yielding a lower bound of $2\pi/\sqrt{3}$

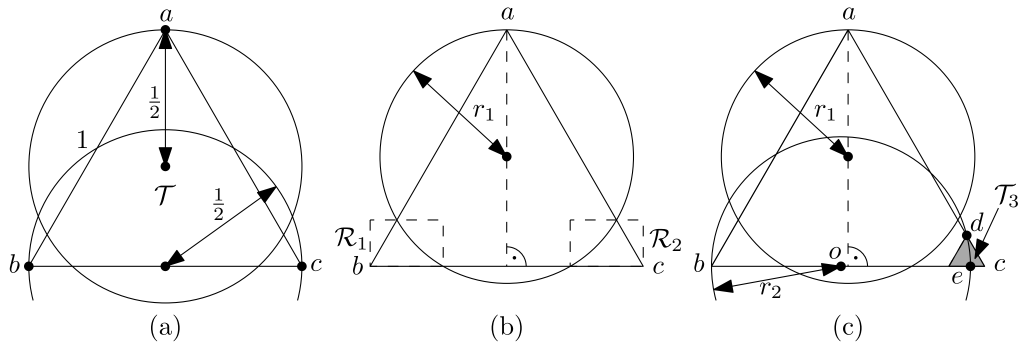

Figure 5 (a) The worst-case for covering equilateral triangles, consisting of two disks of radius $1/2$. (b) Case (1.1), (c) Case (1.3).

on the critical covering coefficient. In the following, we sketch the recursive algorithm that provides the matching upper bound.

Recall that we assume w.l.o.g. that the given set of disks D has exactly the critical covering area $\pi/2$. The radius of a single disk that is sufficient to cover the triangle $\mathcal{T}$ is $\rho(\pi/3) = 1/\sqrt{3}$. If $r_1 \geq \rho(\pi/3)$, the largest disk suffices to cover the entire triangle. If $D = \{r_1\}$, $r_1 = 1/\sqrt{2} > \rho(\pi/3)$; this is the induction base of our proof.

If $\rho(\pi/3) > r_1 \geq 1/2$, we place r_1 covering a trapezoid $\mathcal{R}(\pi/3, r_1)$ including a side of $\mathcal{T}$ as depicted in Figure 1(b), leaving an equilateral triangle $\mathcal{T}_1$ uncovered. Lemma 7 guarantees that placing r_1 in that way covers more triangle area per disk area than covering $\mathcal{T}$ with a single disk of radius $\rho(\pi/3)$, which requires $\frac{\pi \tan(\pi/6)}{\sin^2(\pi/3)}$ units of disk area per unit of triangle area. Thus, because $\frac{\pi \tan(\pi/6)}{\sin^2(\pi/3)} < \frac{2\pi}{\sqrt{3}}$, Lemma 7 guarantees that the disk area that remains after placing r_1 in this manner is large enough to inductively apply Theorem 2 to the remaining equilateral triangle. Note that this includes the case $D = \{r_1, r_2\}$, because $r_1 \geq 1/2$ must hold in that case.

For $r_1 < \frac{1}{2}$, we distinguish the four cases (1) $r_1 > \frac{3\sqrt{3}+\sqrt{10}}{17}$, (2) $\frac{3\sqrt{3}+\sqrt{10}}{17} \geq r_1 > \frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}$, (3) $\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}} \geq r_1 > \frac{48-22\sqrt{3}}{39\sqrt{2}}$, and (4) $\frac{48-22\sqrt{3}}{39\sqrt{2}} \geq r_1$. In the following, we describe how our algorithm handles each case; in Section 4.2, we prove that this always results in a successful cover.

Case (1) We place the largest disk r_1 at distance r_1 below a, see Figure 5(b). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two congruent smallest rectangles containing the remaining uncovered pockets. We distinguish the following subcases.

- **Case (1.1):** $r_2 < 0.11114$: We subdivide D into two subsets D_1, D_2 using Greedy Splitting and then apply our algorithm for covering rectangles to each $\mathcal{R}_i$ with disks D_i, see Figure 5 (b).
- **Case (1.2):** $0.11114 \leq r_2 \leq 0.47186$: We cover $\mathcal{R}_1$ by r_2 and recurse on $\mathcal{R}_2$ with $D \setminus \{r_1, r_2\}$.
- **Case (1.3):** $r_2 > 0.47186$: We place the center of r_2 at distance r_2 to the right of b; see Figure 5(c). Let $\mathcal{T}_3$ be the smallest right-aligned equilateral triangle containing the uncovered area. We recurse on $\mathcal{T}_3$ with disks $D \setminus \{r_1, r_2\}$.

Case (2) We place r_1 at distance r_1 below a and apply our rectangle covering algorithm to the bounding box of the uncovered area with the remaining disks; see Figure 6(a).

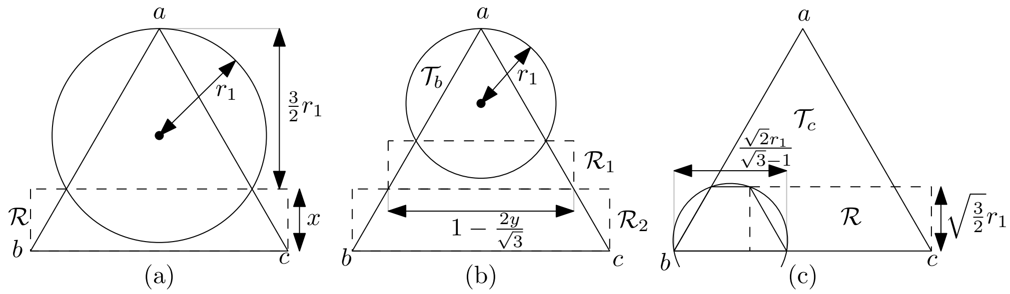

Figure 6 The covering our algorithm computes for (a) Case (2), (b) Case (3), (c) Case (4).

Case (3) We place r_1 at distance r_1 below a, covering an equilateral triangle $\mathcal{T}_b$ at the top of $\mathcal{T}$; see Figure 6(b). Furthermore, we construct two non-overlapping rectangles $\mathcal{R}_1, \mathcal{R}_2$ covering the remaining uncovered isosceles trapezoid. We choose the height y of the lower rectangle $\mathcal{R}_2$ as follows: We iteratively assign the disks from D to $\mathcal{R}_2$ until the assigned weight w_2 exceeds the value $\frac{11}{12}\frac{6}{\sqrt{77}}\left(\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}\right) \approx 0.16588$. We then set $y = {}^{12w_2}/_{11}$. The height x of $\mathcal{R}_1$ is induced by the height of $\mathcal{R}_2$. We assign the remaining unassigned weight to $\mathcal{R}_1$ and apply our rectangle covering algorithm to the rectangles using the assigned disks.

Case (4) We place r_1 covering an isosceles trapezoid with height $\sqrt{3r_1^2/2}$ and baseline length $\frac{\sqrt{2}r_1}{\sqrt{3}-1}$ in the lower left corner of $\mathcal{T}$. We cover the remaining area by another equilateral triangle $\mathcal{T}_c$ and a rectangle $\mathcal{R}$ as illustrated in Figure 6(c). We then iteratively assign the remaining disks in non-increasing order to $\mathcal{T}_c$ until the assigned weight w_c exceeds ${}^{2\|\mathcal{T}_c\|}/_{\sqrt{3}}$. The remaining disks are assigned to $\mathcal{R}$. We then apply our rectangle covering algorithm to $\mathcal{R}$ and recurse on $\mathcal{T}_c$ with the assigned disks.

3.3 Covering obtuse isoceles triangles

In this section, we sketch the proof of Theorem 3; details can be found in Section 4.3.

▶ **Theorem 3.** *For $\alpha \geq \pi/2$, the critical covering coefficient for an isosceles triangle with apex angle α is $\pi \tan(\alpha/2)$.*

The worst-case instance consists of a single disk r_1 whose diameter is the length of the hypotenuse τ of our triangle $\mathcal{T}$; any smaller single disk cannot cover the triangle. The area of $\mathcal{T}$ is $\|\mathcal{T}\| = \frac{\tau^2}{4\tan(\alpha/2)}$, the area of r_1 is $\frac{\pi\tau^2}{4}$, yielding a lower bound of $\pi \tan(\alpha/2)$ on the critical covering coefficient.

In the following, we assume that the hypotenuse of $\mathcal{T}$ has length $\sqrt{2}$ and is horizontal; all other cases can be handled by scaling and rotating. We reduce the case of $\alpha > \pi/2$ to the case $\alpha = \pi/2$, as depicted in Figure 7(a), by covering the smallest right-angled isosceles triangle $\mathcal{T}' \supseteq \mathcal{T}$ instead of $\mathcal{T}$, for which we have exactly the same critical covering area. If the covering computed for $\mathcal{T}'$ contains disks whose centers are outside of $\mathcal{T}$, these can be moved to the boundary of $\mathcal{T}$ without increasing the distance from the center to any point of $\mathcal{T}$; see Figure 7(a).

Therefore, we restrict our attention to right-angled isosceles triangles, which we also call *halfsquares*. Thus we can assume without loss of generality that the total weight of our collection D of disks is $1/2$. We present a recursive algorithm that provides an upper bound

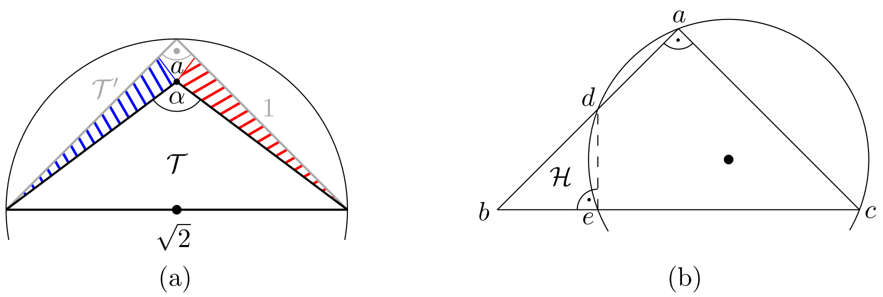

Figure 7 (a) Reducing the case of obtuse isosceles triangles to right-angled ones. If the covering C of $\mathcal{T}'$ contains disks centered outside $\mathcal{T}$, we can move them to $\mathcal{T}$ by orthogonally projecting them onto the closer side of $\mathcal{T}$ if they are in the area shaded in blue or red, and by moving them to the apex a if they are in the unshaded region above a. (b) The situation considered in Lemma 8.

on the critical covering coefficient matching the lower bound shown above. A key ingredient in our proof is the following lemma that shows that the triangle $\mathcal{H}$ that remains after placing a disk as depicted in Figure 7(b) is a halfsquare.

▶ **Lemma 8.** *The triangle $\mathcal{H}$ left after placing a disk r such that its boundary contains the endpoints of a short side of $\mathcal{T}$ is a halfsquare.*

We distinguish several cases based on the two largest disks r_1, r_2. For the case $D = \{r_1, r_2\}$, we have a lemma that guarantees that covering is possible; essentially, we prove that placing r_1 as depicted in Figure 7(b) is possible and efficient enough to allow covering the remaining halfsquare by r_2. Together with the case $D = \{r_1\}$, in which r_1 suffices to cover $\mathcal{T}$, this forms our induction base.

If there are at least three disks, we distinguish five cases based on r_1 and r_2: (1) $\frac{1}{\sqrt{2}} > r_1 \geq \frac{1}{2}$, (2) $\frac{1}{2} > r_1 \geq \frac{1}{2\sqrt{2}}$ and $r_1 + r_2 < \frac{1}{\sqrt{2}}$, (3) $\frac{1}{2} > r_1 \geq \frac{1}{2\sqrt{2}}$ and $r_1 + r_2 \geq \frac{1}{\sqrt{2}}$, (4) $r_1 < \frac{1}{2\sqrt{2}}$ and $r_1 + r_2 \geq \frac{5}{6\sqrt{2}}$, and (5) $r_1 < \frac{1}{2\sqrt{2}}$ and $r_1 + r_2 < \frac{5}{6\sqrt{2}}$. In the following, we describe how our algorithm handles each case; we prove that this always results in a successful covering in Section 4.3.

Case (1) We place r_1 containing the endpoints of a short side of $\mathcal{T}$ on its boundary as depicted in Figure 7(b), which works because $r_1 \geq \frac{1}{2}$. Let $\mathcal{H}$ be the uncovered triangle after placing r_1. Lemma 8 implies that $\mathcal{H}$ is a halfsquare. We recurse on $\mathcal{H}$ with the remaining weight $\frac{1}{2} - r_1^2$.

Case (2) We place r_1 holding the apex on its boundary with its center on the angular bisector of the apex angle, see Figure 8(a). Let D_1, D_2 be a partition of D resulting from an application of Greedy Splitting to D. Furthermore, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the smallest halfsquares that cover the uncovered pockets after placing r_1 as depicted in Figure 8(a). We separately recurse on $\mathcal{H}_1$ and $\mathcal{H}_2$ with D_1 and D_2.

Case (3) We place the centers m_1, m_2 of r_1, r_2 with a distance of $\sqrt{r_1^2 - \left(\frac{r_1^2 - r_2^2 + 1/2}{\sqrt{2}}\right)^2}$ to the hypotenuse such that the distances between a and m_1 and between m_2 and c along ac are $\frac{r_1^2 - r_2^2 + 1/2}{\sqrt{2}}$ and $\frac{r_2^2 - r_1^2 + 1/2}{\sqrt{2}}$, see Figure 8(b). Let $\mathcal{R}$ be the rectangle induced by b and the lower intersection point e between the boundaries of r_1 and r_2, see the gray area in Figure 8(b). We apply our rectangle covering algorithm on the rectangle $\mathcal{R}$ with the remaining disks.

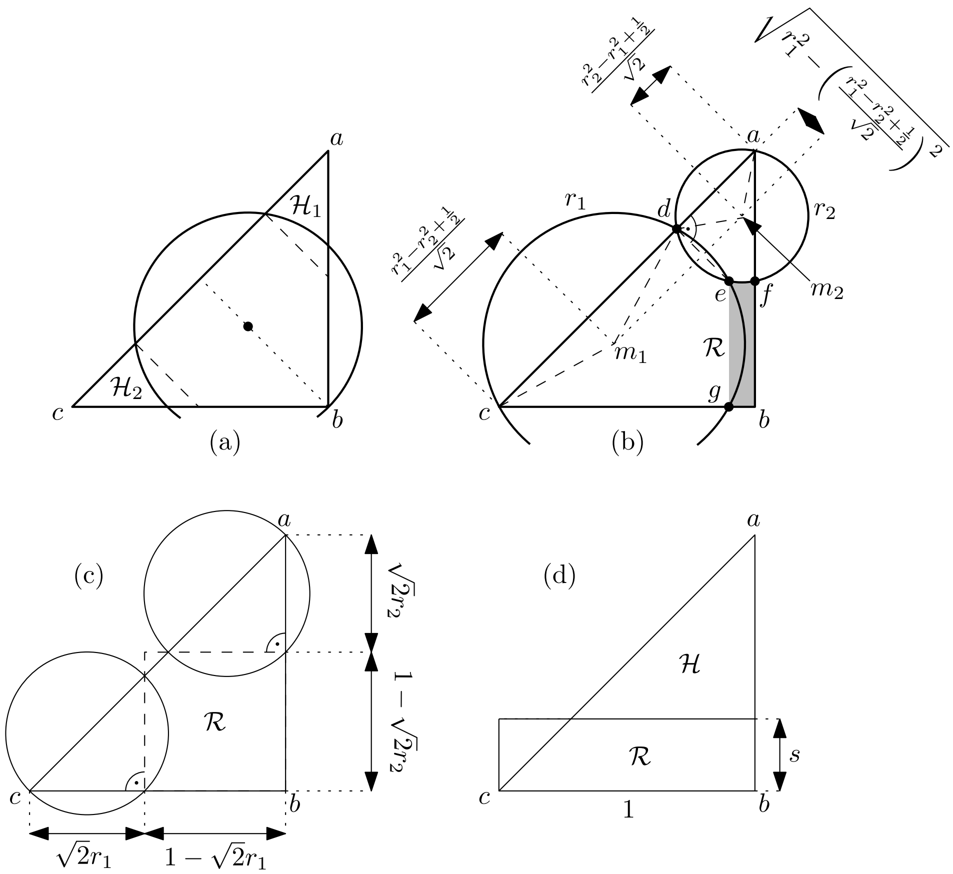

Figure 8 (a) Placement of r_1 in Case (2). (b) Placement of r_1 and r_2 in Case (3). (c) Placement of r_1, r_2 and the rectangle $\mathcal{R}$ in Case (4). (d) Placement of the strip $\mathcal{R}$ and the remaining halfsquare $\mathcal{H}$ in Case (5).

Case (4) We place the largest disk r_1 holding c on its boundary and with its midpoint on the hypotenuse ac, see Figure 8(c). Analogously, we place the second largest disk r_2 holding a on its boundary and with its midpoint on the hypotenuse ac. Finally, we recurse on the smallest rectangle $\mathcal{R}$ containing the uncovered area after placing r_1, r_2 with the remaining disks.

Case (5) We build a subset of disks $D_s \subset D$ and use it to cover a rectangular strip $\mathcal{R}$ of some height s including the bottom side of our triangle, see Figure 8(d). From Lemma 5, recall $\hat{\sigma} := {}^{195\sqrt{5257}}/_{16384}$. We start building D_s by adding disks in non-increasing order of weight, starting with r_2, until the collected weight $w(D_s)$ is at least $w(D_s) \geq \frac{195r_2}{256\sqrt{\hat{\sigma}}}$. We set $s = \frac{256w(D_s)}{195}$ and use Lemma 5 to cover the rectangular strip $\mathcal{R}$ with the disks from D_s, and recurse on the remaining half-square $\mathcal{H}$ using the remaining disks $D \setminus D_s$.

4 Details

In this section, we present the details of the longer proofs that we omitted Section 3 for better readability.

4.1 Details for small apex angle

In this section, we present the remaining details for covering isoceles triangles with small apex angle; for a high-level overview, refer to Section 3.1.

4.1.1 Details for R1 center

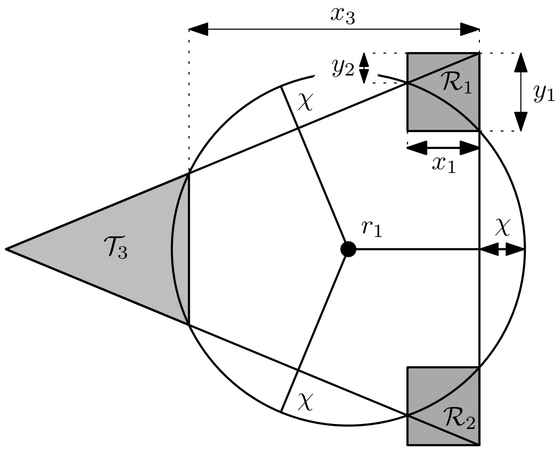

Figure 9 Placement of r_1 by routine R1 CENTER and the three remaining pockets $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{T}_3$ and their sizes.

Recall that R1 CENTER places the disk r_1 on the vertical center line of the triangle, such that the distance χ that the disk protrudes from the triangle is the same on all three sides; see Figure 9. In that case, we have

$$\chi = \frac{r_1 \sin\left(\frac{\alpha}{2}\right) + r_1 - \frac{\cos\left(\frac{\alpha}{2}\right)}{2}}{\sin\left(\frac{\alpha}{2}\right) + 1}.$$

If $\chi \geq r_1$, we cannot place r_1 as desired and the covering routine fails.

Otherwise, we can guarantee that we can place r_1 in the desired way and that the boundary of r_1 has two intersection points with all three sides of $\mathcal{T}$. This leaves uncovered two pockets at the base, for which we cover their bounding boxes, and a triangle $\mathcal{T}_3$. We compute the intersections points and can thus determine the size of the three remaining pockets; see Figure 9. Let

$$c_1 := \sqrt{\frac{16r_1^2 \sin\left(\frac{\alpha}{2}\right) + 4r_1^2 \cos(\alpha) \tan^2\left(\frac{\alpha}{2}\right) + 4r_1^2 \tan^2\left(\frac{\alpha}{2}\right) + 8r_1^2 - \cos(\alpha) - 1}{\cos(\alpha) + 1}},$$

then we compute the sizes of the remaining pockets as follows:

$$x_1 = \frac{\left(1 - \sin\left(\frac{\alpha}{2}\right)\right)\left(-c_1 \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) + 1\right)}{2\cos\left(\frac{\alpha}{2}\right)},$$

$$x_3 = \frac{\left(1 - \sin\left(\frac{\alpha}{2}\right)\right)\left(c_1 \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) + 1\right)}{2\cos\left(\frac{\alpha}{2}\right)},$$

$$y_1 = \frac{1 - \sqrt{8r_1^2 \sin\left(\frac{\alpha}{2}\right) - 4r_1^2 \cos^2\left(\frac{\alpha}{2}\right) + 8r_1^2 - \cos^2\left(\frac{\alpha}{2}\right)} + \sin\left(\frac{\alpha}{2}\right)}{2\left(\sin\left(\frac{\alpha}{2}\right) + 1\right)}, \text{ and}$$

$y_2 = x_1 \cdot \tan(\alpha/2)$.

We then check whether r_2 and r_3 can cover $\mathcal{R}_1, \mathcal{R}_2$ or $\mathcal{T}_3$ on their own. We try covering each possible combination of $\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{T}_3\}$ using r_2, r_3. Between 1 and 3 pockets remain uncovered.

If one pocket remains uncovered, we check whether the remaining disk weight suffices to cover it by recursion (i.e., inductive application of Theorem 1 for $\mathcal{T}_3$), or by our rectangle covering algorithm. This can be done in the success criterion as well because it only depends on α and r_1, r_2, r_3.

If two or three pockets remain uncovered, we have to partition the remaining disks before we can apply rectangle covering or recursion. We do this by iteratively collecting disks until their total weight is sufficient; in our success criterion, we bound the total weight assigned to a pocket P by $w(P) + r_{\max}^2$, where $w(P)$ is the weight required for recursion or rectangle covering and $r_{\max}$ is the size of the largest disk (r_2 or r_3) that may be assigned to P. This allows us to decide whether any of our rectangle covering results or an inductive application of Theorem 1 guarantees success.

4.1.2 Details for two large disks

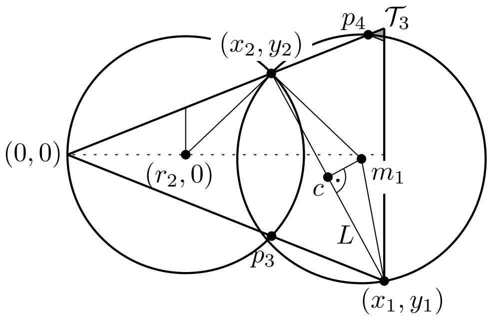

Figure 10 Our covering routine for the case of two large disks $r_1, r_2 \leq 1/2$.

Our routine Two Large Disks works as depicted in Figure 10. The origin of our coordinate system is placed at the apex of the input triangle $\mathcal{T}$. The smaller disk r_2 is placed at distance r_2 to its right. We draw a line segment L through the upper intersection point of r_2 and the upper side of $\mathcal{T}$ and the bottom right corner of $\mathcal{T}$. We place the center m_1 of r_1 such that it covers the line segment L, touching the end points of L with its boundary. If the lower intersection p_3 between r_2 and the bottom side of $\mathcal{T}$ is not inside r_1, Two Large Disks fails. Otherwise, the remaining region is covered by a triangle $\mathcal{T}_3$, on which we recurse.

Let

$$t_1 = \sqrt{\frac{16r_1^2\tan^2\left(\frac{\alpha}{2}\right) - \left(2r_2\sin\left(\alpha\right)+1\right)^2\tan^2\left(\frac{\alpha}{2}\right) - \left(2r_2\left(\cos\left(\alpha\right)+1\right)\tan\left(\frac{\alpha}{2}\right)-1\right)^2}{\left(2r_2\sin\left(\alpha\right)+1\right)^2\tan^2\left(\frac{\alpha}{2}\right) + \left(2r_2\left(\cos\left(\alpha\right)+1\right)\tan\left(\frac{\alpha}{2}\right)-1\right)^2}},$$

then the resulting coordinates for the center m_1 of r_1 are

$$m_x = \frac{\left(2r_2\left(\cos\left(\alpha\right)+1\right) + t_1\left(2r_2\sin\left(\alpha\right)+1\right)\right)\tan\left(\frac{\alpha}{2}\right)+1}{4\tan\left(\frac{\alpha}{2}\right)}, \text{ and}$$

$$m_y = \frac{t_1\left(-2r_2\left(\cos\left(\alpha\right)+1\right)\tan\left(\frac{\alpha}{2}\right)+1\right) + \left(2r_2\sin\left(\alpha\right)-1\right)\tan\left(\frac{\alpha}{2}\right)}{4\tan\left(\frac{\alpha}{2}\right)}.$$

Furthermore, the coordinates (v_x, v_y) of the right intersection point p_4 of r_1 with the top side of $\mathcal{T}$ are

$$v_x = \frac{\sqrt{\frac{8r_1^2\cos(\alpha)-8r_1^2-2r_2^2\cos(2\alpha)+2r_2^2-2r_2\sin(2\alpha)+1}{2r_2^2\cos(2\alpha)-2r_2^2+2r_2\sin(2\alpha)-1}}\sin(\alpha)+\cos(\alpha)}{2\tan\left(\frac{\alpha}{2}\right)}, \text{ and}$$

$$v_y = v_x \tan(\alpha/2).$$

These formulas allow us to compute the size of the triangle covering the remaining region $\mathcal{T}_3$; if the remaining disks suffice to cover $\mathcal{T}_3$ by an inductive application of Theorem 1, our success criterion is satisfied and our algorithm is guaranteed to produce a valid covering.

Note that, for $\alpha = \pi/4$, the case of two disks of radius $r_1 = r_2 = 1/2$ is actually a second worst-case instance with exactly the same total disk area as a single disk covering $\mathcal{T}$. Here, Two Large Disks produces a covering that places r_1 on the center of the base of $\mathcal{T}$. Unfortunately for our interval arithmetic approach, this means that our result is *tight* (in the numerical sense) for α, r_1, r_2 very close to these values. Therefore, the discretization error introduced by using intervals of any positive size implies that our automatic approach does not work for values very close to this two-disk worst case. We introduce the following lemma to address this issue, allowing our computer-assisted approach to avoid the area close to this two-disk worst-case.

▶ **Lemma 9.** *For any $\alpha \in [44°, 45°]$ and any $r_2, r_1 \in [0.48, 0.5]^2$ with $r_2 \le r_1$, Two Large Disks produces a valid covering of the input triangle $\mathcal{T}$.*

Proof. Let α, r_1, r_2 be in the specified ranges. For values in this range, placing r_1 and r_2 as depicted in Figure 10 is always possible. Furthermore, p_3 is always covered by r_1, and the remaining triangle is always induced by the point p_4, instead of the intersection of r_1 with the base.

Let $w_{12} = \frac{\tan(\alpha/2)}{\sin^2(\alpha)} - r_1^2 - r_2^2$ be the weight of the disks that remains after placing r_1, r_2. By w_r, we denote the weight required by Theorem 1 to guarantee a successful covering of $\mathcal{T}_3$. We have

$$w_r = \frac{\left(\sqrt{\frac{8r_1^2\cos(\alpha)-8r_1^2-2r_2^2\cos(2\alpha)+2r_2^2-2r_2\sin(2\alpha)+1}{2r_2^2\cos(2\alpha)-2r_2^2+2r_2\sin(2\alpha)-1}} - \tan\left(\frac{\alpha}{2}\right)\right)^2}{4}.$$

By $\Delta = w_{12} - w_r$, we denote the amount of disk weight by which we exceed the amount required for recursion, and show that $\Delta \ge 0$ for all points we have to consider.

We compute the partial derivative of Δ by r_1 and prove by interval arithmetic that it is negative for all points in our ranges. Therefore, independent of r_2 and α, we can always reduce Δ by increasing r_1 to its maximum $1/2$. Similarly, we then compute the partial derivative of Δ by r_2 for $r_1 = 1/2$; again, we can prove it is negative for all points independent of α. Finally, the partial derivative for α is also negative for $r_2 = r_1 = 1/2$; thus, Δ takes on its global minimum at $\alpha = \pi/4, r_1 = r_2 = 1/2$, where we have $\Delta = 0$. Therefore, the amount of weight by which we exceed the required weight is never negative, implying a successful cover for every point we have to consider. ◀

4.2 Details for equilateral triangles

In this section, we present the details for covering equilateral triangles; for a high-level overview, refer to Section 3.2.

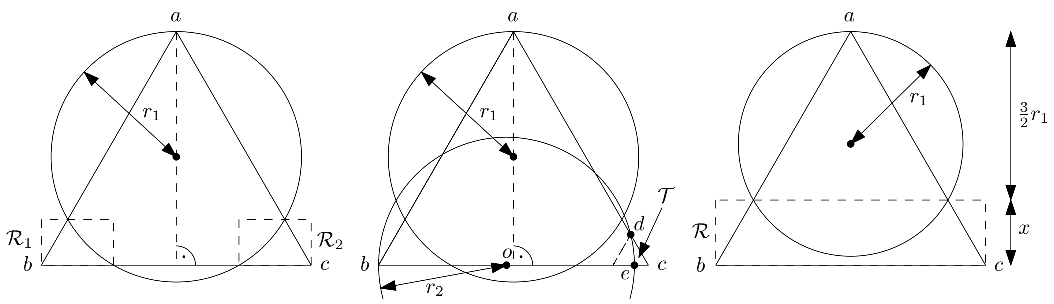

Figure 11 Left: Case (1.1), Middle: Case (1.3), Right: Case (2).

4.2.1 Details for Case (1)

To handle Case (1) (see Figure 11), we show the following lemma.

▶ **Lemma 10.** *The covering routine we use in Case (1) achieves a covering of the entire input triangle $\mathcal{T}$.*

Proof. We observe that the area not covered by the largest disk is made up of two disjoint pockets because the diameter of the largest disk is $2r_1 > 0.98334 \approx 2 \cdot \left(\frac{3\sqrt{3}+\sqrt{10}}{17}\right) > \frac{\sqrt{3}}{2} \approx 0.86602$. We show the lemma for Cases (1.1), (1.2), and (1.3) separately.

- $r_2 < \frac{1}{2}\sqrt{0.1286^2 + 0.1813^2} < 0.11114$: The height of $\mathcal{R}_1$ is $\frac{\sqrt{3}}{2}\left(1-\sqrt{3}r_1\right) \leq 0.1286$, see Figure 11 left. Furthermore, the width of $\mathcal{R}_1$ is $\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}} \leq 0.1813$. Thus, $|\mathcal{R}_1| \leq 0.02331$ and the skew of $\mathcal{R}_1$ is at most 1.5622. Analogously, we get $|\mathcal{R}_2| \leq 0.02331$ and that the skew of $\mathcal{R}_2$ is at most 1.5622. This yields that the weights assigned to $\mathcal{R}_1, \mathcal{R}_2$ are at least as large as $\frac{11}{12}|\mathcal{R}_1|$, $\frac{11}{12}|\mathcal{R}_2|$. In particular, we have:

$$\frac{\frac{1}{2}-r_1^2}{2} - \frac{r_2^2}{2} > \frac{\frac{1}{2}-\frac{1}{4}}{2} - \frac{0.1111^2}{2} \geq 0.06945 > 0.02137 \approx \frac{11}{12}0.02331$$

  because $r_1 \leq \frac{1}{2}$ and $r_2 < 0.1111$. Finally, Theorem 4 implies that the weights assigned to $\mathcal{R}_1, \mathcal{R}_2$ permit coverings of $\mathcal{R}_1, \mathcal{R}_2$.
- $0.11114 \leq r_2 \leq 0.47186$: We have

$$\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}} \leq \frac{3\sqrt{3}}{2}\left(1-\sqrt{3}r_1\right) \tag{2}$$

  because the left side of Inequality 2 is monotonically decreasing and thus attains its maximum ≈ 0.181257 for $r_1 = 0.49167 \approx \frac{3\sqrt{3}+\sqrt{10}}{17}$. Furthermore, the right side is monotonically decreasing and thus attains its minimum ≈ 0.3481 at $r_1 = \frac{1}{2}$. This implies

$$\frac{1}{3} \leq \frac{\frac{\sqrt{3}}{2}\left(1-\sqrt{3}r_1\right)}{\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}}}. \tag{3}$$

  Furthermore, we have

$$\frac{\sqrt{3}}{2}\left(1-\sqrt{3}r_1\right) \leq \frac{3}{2} - 3\sqrt{\sqrt{3}r_1 - \frac{3}{4}} \tag{4}$$

  because the left side of Inequality 4 attains its maximum ≈ 0.12852 at $r_1 = 0.49167 \approx \frac{3\sqrt{3}+\sqrt{10}}{17}$ and the right side attains its minimum ≈ 0.478124 at $r_1 = \frac{1}{2}$. This implies

$$\frac{\frac{\sqrt{3}}{2}\left(1-\sqrt{3}r_1\right)}{\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}}} \leq 3. \tag{5}$$

Combining Inequalities 3 and 5 yields that $\mathcal{R}_1$ has a skew λ of at most 3. We have $r_2^2 \leq 0.22865$ because $r_2 \leq 0.47186$. Thus, we upper bound r_2^2 as follows:

$$\begin{aligned} r_2^2 &\leq 0.22865 \\ &\leq \frac{1}{4} - \frac{11\sqrt{3}}{24} \cdot 0.14841 \cdot 0.18126 \\ &\leq \underbrace{\frac{1}{2} - r_1^2}_{\substack{\geq \frac{1}{4} \\ \text{as } r \leq \frac{1}{2}}} - \frac{11\sqrt{3}}{24} \underbrace{\left(1 - \sqrt{3}r_1\right)}_{\substack{\leq 0.14841 \\ \text{as } r_1 \leq 0.49168}} \underbrace{\left(\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}}\right)}_{\substack{\leq 0.18126 \\ \text{as } r_1 \leq 0.49168}} \end{aligned} \quad (6)$$

$$(7)$$

Inequality 6 is equivalent to

$$\begin{aligned} \frac{1}{2} - r_1^2 - r_2^2 &\geq \frac{11}{12} \cdot \frac{\sqrt{3}}{2} \left(1 - \sqrt{3}r_1\right) \left(\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}}\right) \\ &\geq \frac{\lambda^2 + 2}{2} \cdot \frac{\sqrt{3}}{2} \left(1 - \sqrt{3}r_1\right) \left(\frac{1}{2} - \sqrt{\sqrt{3}r_1 - \frac{3}{4}}\right). \end{aligned}$$

Thus, Theorem 4 implies that the weight assigned to $\mathcal{R}_2$ permits a covering of $\mathcal{R}_2$ because the skew λ of $\mathcal{R}_1$ is at most 3.

- $r_2 > 0.47186$: Let ε such that $r_2 = \frac{1}{2} - \varepsilon$, e the second intersection of the boundary of r_2 with the base line bc, and d the intersection of the boundary of r_2 with the ac which lies closer to c, see Figure 11 right. We have $|ec| = |bc| - |be| = 1 - 2r_2 = 2\varepsilon$.
Let $o \in bc$ be the midpoint of r_2. Using the cosine rule, we obtain $\cos\left(\angle(d, c, o)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = \frac{|dc|^2 + (1-r_2)^2 - r_2^2}{2x(1-r_2)}$. Thus we obtain

$$\begin{aligned} & |dc|^2 + |dc|(r_2 - 1) + 1 - 2r_2 = 0 \\ \Rightarrow \quad & |dc| = \frac{1}{4} + \frac{\varepsilon}{2} - \frac{1}{4}\sqrt{1 - 4\left(7\varepsilon - \varepsilon^2\right)} \\ \Rightarrow \quad & |dc| < \frac{1}{2}\left(15\varepsilon - 2\varepsilon^2\right) \\ & \quad < \frac{15\varepsilon}{2}, \end{aligned}$$

because $1 - \delta \leq \sqrt{1 - \delta}$ for $\delta \in [0, 1]$.
We have $r_2 > 0.47186 > \frac{1}{2} - \frac{8}{233}$ which implies $\varepsilon \leq \frac{8}{233}$. Thus, we obtain

$$\begin{aligned} & \frac{8}{233} \geq \varepsilon \\ \Leftrightarrow \quad & 1 \geq \frac{233}{8}\varepsilon \\ \Leftrightarrow \quad & \varepsilon \geq \frac{233}{8}\varepsilon^2 \\ \Leftrightarrow \quad & \frac{1}{2} - \frac{1}{4} - \frac{1}{4} - \varepsilon^2 + \varepsilon \geq \frac{225}{8}\varepsilon^2 \\ \Rightarrow \quad & \frac{1}{2} - r_1^2 - r_2^2 \geq \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{4}\left(\frac{15}{2}\varepsilon\right)^2. \end{aligned}$$

This means that the remaining weight $\frac{1}{2} - r_1^2 - r_2^2$ is at least $\frac{2}{\sqrt{3}}$ times the area of an right-aligned equilateral triangle with side length $\frac{15\varepsilon}{2}$. As the side length $|dc|$ of $\mathcal{T}$ is smaller than $\frac{15\varepsilon}{2}$, we obtain that the remaining weight permits a covering of $\mathcal{T}$. This concludes the proof. ◀

4.2.2 Details for Case (2)

In Case (2), we have to prove that we can cover the remaining rectangle $\mathcal{R}$ using our rectangle covering algorithm, see Figure 11 right.

▶ **Lemma 11.** *In Case (2), the remaining weight $w(D \setminus \{r_1\})$ permits a covering of $\mathcal{R}$.*

Proof. Note that in this case we already have $\frac{3\sqrt{3}+\sqrt{10}}{17} \geq r_1 > \frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}$. Furthermore, for the entire analysis of Case (2) we distinguish between two (sub) cases regarding r_1:

- $\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}} \leq r_1 \leq \frac{1}{\sqrt{3}} - \frac{2}{9}$: From $r_1 \leq \frac{1}{\sqrt{3}} - \frac{2}{9}$ we obtain $r_1 \leq \frac{2}{3}\left(\frac{\sqrt{3}}{2} - \frac{1}{3}\right)$ which is equivalent to $1 \leq 3\frac{\sqrt{3}-3r_1}{2}$. This implies that $\mathcal{R}$ has a skew λ of at most 3 because $\mathcal{R}$ has a width of 1 and a height of $\frac{\sqrt{3}-3r_1}{2}$.
  Next, we lower bound the remaining weight $\frac{1}{2} - r_1^2$ by $\frac{11}{12}\frac{\sqrt{3}-3r_1}{2}$. Based on that, Theorem 4 implies that the remaining weight permits a covering of $\mathcal{R}$ because $\lambda := \frac{\sqrt{3}-3r_1}{2} \leq 3$ implies $\frac{11\lambda}{12} \geq \frac{\lambda^2+2}{4}$.
  From $r_1 \leq \frac{1}{\sqrt{3}} - \frac{2}{9} \approx 0.35512$ we obtain $r_1 \leq \frac{11}{16} + \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}} \approx 1.1103$. Thus, we have

$$\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}} \leq r_1 \leq \frac{11}{16} + \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}$$

$$\Leftrightarrow 0 \geq r_1^2 - \frac{11r_1}{8} + \frac{11\sqrt{3}}{24} - \frac{1}{2} \Leftrightarrow \frac{1}{2} - r_1^2 \geq \frac{11}{12}\left(\frac{\sqrt{3}-3r_1}{2}\right).$$

- $\frac{1}{\sqrt{3}} - \frac{2}{9} < r_1 \leq \frac{3\sqrt{3}+\sqrt{10}}{17}$: From $0.35512 \approx \frac{1}{\sqrt{3}} - \frac{2}{9} < r_1$, we obtain $0.11964 \approx \frac{3\sqrt{3}-\sqrt{10}}{17} \leq r_1$. Thus we have

$$\frac{3\sqrt{3}-\sqrt{10}}{17} \leq r_1 \leq \frac{3\sqrt{3}+\sqrt{10}}{17}$$

$$\Leftrightarrow \frac{17r_1^2}{8} - \frac{3\sqrt{3}r_1}{4} + \frac{1}{8} \leq 0 \Leftrightarrow \frac{1}{2} - r_1^2 \geq \frac{1 + 2\left(\frac{\sqrt{3}-3r_1}{2}\right)^2}{4}.$$

  Hence, the remaining weight $\frac{1}{2} - r_1^2$ is at least $\frac{1+2x^2}{4} \cdot \frac{1}{x^2}$ with $x := \frac{\sqrt{3}-3r_1}{2}$, which is exactly the weight required by Theorem 4 for an $x \times 1$-rectangle with $x \leq 1/\lambda_2$. Thus, Theorem 4 implies that $\mathcal{R}$ can be covered by the remaining weight. This concludes the proof. ◀

4.2.3 Details for Case (3)

In Case (3), we have to show that the disks assigned to the two remaining rectangles are sufficient to cover them; see Figure 6(b).

▶ **Lemma 12.** *The disks assigned to $\mathcal{R}_1$ and $\mathcal{R}_2$ permit coverings of $\mathcal{R}_1$ and $\mathcal{R}_2$.*

Proof. By construction, the weight assigned to $\mathcal{R}_2$ is equal to $\frac{11}{12}|\mathcal{R}_2|$ and the radius of the largest disk assigned to $\mathcal{R}_2$ is at most $0.2646 < \frac{\sqrt{77}}{6} = \sqrt{\sigma\left(\frac{11}{12}\right)}$. Therefore, Lemma 5 implies that the weight assigned to $\mathcal{R}_2$ permits a covering of $\mathcal{R}_2$.

The height y of the lower rectangle $\mathcal{R}_2$ is given as follows:

$$y = \frac{6}{\sqrt{77}}\left(\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}\right) - \frac{12\delta}{11}. \tag{8}$$

for $\delta \in [0, r_1^2]$. Furthermore, the height x of the upper rectangle $\mathcal{R}_1$ is given as $x = \frac{\sqrt{3}}{2} - y - \frac{3r_1}{2}$. Applying Equality 8 yields:

$$x = \frac{\sqrt{3}}{2} - \frac{3r_1}{2} - \frac{6}{\sqrt{77}}\left(\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}\right) + \frac{12\delta}{11}.$$

The width of $\mathcal{R}_1$ is $w = 1 - \frac{2y}{\sqrt{3}}$. Consider the skew $\lambda := \max\{\frac{x}{w}, \frac{w}{x}\}$ of $\mathcal{R}_1$. Thus, we have everything at hand to check prove via interval arithmetic that the remaining weight is large enough to cover $\mathcal{R}_1$. In particular, we prove that

$$\frac{1}{2} - r_1^2 - \underbrace{\frac{11}{12}\frac{6}{\sqrt{77}}\left(\frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}\right) - \delta}_{\text{Weight used to cover } \mathcal{R}_2}$$

suffices to cover $\mathcal{R}_1$ by Theorem 4 for all $\delta \in [0, r_1^2]$ and $r_1 \in \left[\frac{48-22\sqrt{3}}{39\sqrt{2}}, \frac{11}{16} - \sqrt{\frac{249}{256} - \frac{11\sqrt{3}}{24}}\right]$. Thus, Theorem 4 implies that the remaining weight permits a covering of $\mathcal{R}_1$. ◀

4.2.4 Details for Case (4)

In Case (4), we have to show that the disks assigned to the remaining triangle $\mathcal{T}_c$ and rectangle $\mathcal{R}$; see Figure 6(c).

▶ **Lemma 13.** *The disks assigned to $\mathcal{T}_c, \mathcal{R}$ permit coverings of $\mathcal{T}_c, \mathcal{R}$.*

Proof. Our algorithm itself guarantees that the weight assigned to $\mathcal{T}_c$ is at least $\frac{2\pi}{\sqrt{3}}$ times the area of $\mathcal{T}_c$. Thus, Theorem 2 implies that the weight assigned to $\mathcal{T}_c$ permits covering $\mathcal{T}_c$. Thus it still remains to be proven that the weight assigned to the rectangle $\mathcal{R}$ permits covering $\mathcal{R}$.

The weight assigned to $\mathcal{T}_c$ is at most $\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{4}(\sqrt{2}r_1)^2 + r_1^2$. Thus, the weight assigned to $\mathcal{R}$ is at least $\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{4} - r_1^2 - \left(\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{4}(\sqrt{2}r_1)^2 + r_1^2\right) = \frac{1}{2} - 3r_1^2$.

The area of $\mathcal{R}$ is $\sqrt{\frac{3}{2}}r_1\left(1 - \frac{\sqrt{2}r_1}{\sqrt{3}-1} + \frac{r_1}{\sqrt{2}}\right)$.

In order to show that the weight assigned to $\mathcal{R}$ is enough to cover $\mathcal{R}$, we consider the inequality

$$\frac{1}{2} \geq \frac{11}{12}\left(\left(\frac{36}{11} - \frac{\sqrt{3}}{\sqrt{3}-1} + \frac{\sqrt{3}}{2}\right)r_1^2 + \sqrt{\frac{3}{2}}r_1\right). \tag{9}$$

We have $\frac{36}{11} - \frac{\sqrt{3}}{\sqrt{3}-1} + \frac{\sqrt{3}}{2} \geq 0$ implying that the right-hand side of Inequation 9 is monotonically increasing in r_1. Hence, we assume $r_1 = \frac{48-22\sqrt{3}}{39\sqrt{2}}$ for which Inequality 9 is true.

We apply the following equivalent transformations:

$$\frac{1}{2} \geq \frac{11}{12}\left(\left(\frac{36}{11} - \frac{\sqrt{3}}{\sqrt{3}-1} + \frac{\sqrt{3}}{2}\right) r_1^2 + \sqrt{\frac{3}{2}} r_1\right) \tag{10}$$

$$\Leftrightarrow \frac{1}{2} - 3r_1^2 \geq \frac{11}{12}\left(\left(-\frac{\sqrt{3}}{\sqrt{3}-1} + \frac{\sqrt{3}}{2}\right) r_1^2 + \sqrt{\frac{3}{2}} r_1\right) \tag{11}$$

$$\Leftrightarrow \frac{1}{2} - 3r_1^2 \geq \frac{11}{12}\left(\sqrt{\frac{3}{2}} r_1 \left(1 - \frac{\sqrt{2} r_1}{\sqrt{3}-1} + \frac{r_1}{\sqrt{2}}\right)\right) \tag{12}$$

Inequality 12 implies that the weight assigned to $\mathcal{R}$ is at least $\frac{11}{12}$ times the area of $\mathcal{R}$. Finally, Lemma 5 implies that the weight assigned to $\mathcal{R}$ permits a covering of $\mathcal{R}$ because $\sqrt{\frac{3}{2}} \approx 1.2247\ldots < \sqrt{\sigma\left(\frac{11}{12}\right)} = \frac{\sqrt{77}}{6} \approx 1.4625\ldots$. ◀

4.3 Details for halfsquares

In this section, we present the details for covering halfsquares, i.e., right-angled isoceles triangles. For a high-level overview, refer to Section 3.3.

4.3.1 Proof of Lemma 8

In this section, we present the proof of Lemma 8.

▶ **Lemma 8.** *The triangle $\mathcal{H}$ left after placing a disk r such that its boundary contains the endpoints of a short side of $\mathcal{T}$ is a halfsquare.*

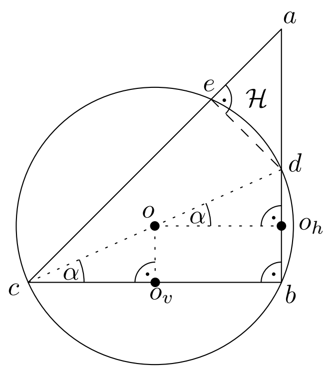

Figure 12 The triangle $\mathcal{H}$ left after placing a disk $r \geq \frac{|bc|}{2}$ holding b, c on its boundary is a halfsquare.

Proof. Let o_v and o_h be the orthogonal projections of o onto bd and bd; see Figure 12. As c and b lie on the boundary of r_1 we obtain $|co_v| = |bo_v| = \frac{1}{2}$. Furthermore, we have $|co| = |do| = r_1$ and $\angle(c, o_v, o) = \angle(o, o_h, d) = \frac{\pi}{2}$. Thus, the triangles $\triangle(c, o_v, o)$ and $\triangle(o, o_h, d)$ are congruent which implies $\angle(o, c, o_v) = \angle(d, o, o_h)$. As co_h and oo_h lie in parallel, we obtain that o lies on cd. This means that cd is the diameter of r_1. By Thales' Theorem we obtain $\angle(c, e, d) = \frac{\pi}{2}$. Thus, $\angle(a, e, d) = \frac{\pi}{2}$. This implies $\angle(e, d, a) = \pi - \frac{\pi}{2} = \frac{\pi}{4} = \frac{\pi}{4}$, which means that $\mathcal{H}$ is a halfsquare. ◀

4.3.2 Details for two disks

Recall that we need to handle the case $D = \{r_1, r_2\}$ in our induction base; to this end, we prove the following lemma.

▶ **Lemma 14.** *Any pair of disks with radii $r_1 \geq r_2$ and total weight $r_1^2 + r_2^2$ at least $1/2$ permits a covering of a halfsquare $\mathcal{H}$ with hypotenuse $\sqrt{2}$.*

Proof. If $r_1 \geq \frac{1}{\sqrt{2}}$, $\mathcal{T}$ is covered by placing the midpoint of r_1 on the midpoint of the hypotenuse. Thus, we assume $r_1 < \frac{1}{\sqrt{2}}$. From $r_1^2 + r_1^2 \geq \frac{1}{2}$, we get $r_1 \geq \frac{1}{2}$ because $r_1 \geq r_2$. This allows that b and c lie on the boundary of r_1.

Lemma 8 implies that the triangle $\mathcal{T}$ induced by the uncovered area after placing r_1 is a halfsquare. Let d be the intersection point of ab with the boundary of r_1, see Figure 12. Thus, $|ad| = \frac{1}{2} - \sqrt{r_1^2 - \frac{1}{4}}$. In order to show that $\mathcal{T}$ can be covered by r_2, we consider the inequality $\sqrt{\frac{1}{2} - r_1^2} \geq \frac{1}{2} - \sqrt{r_1^2 - \frac{1}{4}}$ which is true for $r_1 \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$. Furthermore, $r_1^2 + r_2^2 \geq \frac{1}{2}$ implies $r_2 \geq \sqrt{\frac{1}{2} - r_1^2}$. This yields, $r_2 \geq \frac{1}{2} - \sqrt{r_1^2 - \frac{1}{4}}$, i.e., $2r_2$ is at least as large as the length of the hypotenuse of $\mathcal{H}$ implying that $\mathcal{H}$ can be covered by r_2. ◀

4.3.3 Details for Case (1)

We use the following lemma to handle Case (1); see Figure 7(b).

▶ **Lemma 15.** *The remaining weight $\frac{1}{2} - r_1^2$ permits a covering of the uncovered halfsquare $\mathcal{H}$.*

Proof. By assumption, we have $r_1 < \frac{1}{\sqrt{2}}$ which implies $\sqrt{r_1^2 - \frac{1}{4}} \leq \frac{1}{2}$. We apply the following equivalent transformations:

$$\begin{aligned} \sqrt{r_1^2 - \frac{1}{4}} &\leq \frac{1}{2} \\ \Leftrightarrow \quad r_1^2 - \frac{1}{4} &\leq \frac{1}{2}\sqrt{r_1^2 - \frac{1}{4}} \\ \Leftrightarrow \left(\sqrt{r_1^2 - \frac{1}{4}} - \frac{1}{2}\right)^2 &\leq \frac{1}{2} - r_1^2. \end{aligned}$$

This means that the remaining weight is at least as large as the area of $\mathcal{H}$. Lemma 8 implies that $\mathcal{H}$ is a halfsquare. This concludes the proof. ◀

4.3.4 Details for Case (2)

We use the following lemma to handle Case (2); see Figure 8(a).

▶ **Lemma 16.** *The weights of D_1, D_2 permit coverings of $\mathcal{H}_1, \mathcal{H}_2$.*

Proof. Consider the function $f(r_1) := 4r_1^2 - 3\sqrt{2}r_1 + 1$. The first derivative of f is $8r_1 - 3\sqrt{2}$ and the second derivative of f is positive. Furthermore, we have $f\left(\frac{1}{\sqrt{2}}\right) = 2 - 3 + 1 = 0$ and $f\left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{2} - \frac{3}{2} + 1 = 0$. Thus, we have

$$\begin{aligned} 0 &\geq 4r_1^2 - 3\sqrt{2}r_1 + 1 \\ \Leftrightarrow \frac{1}{2}\left(\sqrt{2}r_1 - 2r_1^2\right) &\geq \frac{1}{2}\left(1 - \sqrt{2}r_1\right)^2 \\ \overset{(\star)}{\Rightarrow} \quad \frac{1}{4} - \frac{r_1^2 + r_2^2}{2} &\geq \frac{1}{2}\left(1 - \sqrt{2}r_1\right)^2. \end{aligned}$$

Implication (⋆) applies $r_2 \leq \frac{1}{\sqrt{2}} - r_1$ which is guaranteed by greedy splitting. This means that weight of D_1 is at least as large as the area of $\mathcal{H}_1$.

Analogously, we obtain that the weight of D_2 is at least as large as the area of $\mathcal{H}_2$. This concludes the proof. ◀

4.3.5 Details for Case (3)

We use the following lemma to handle Case (3); see Figure 8(b). For Case (3), we prove that

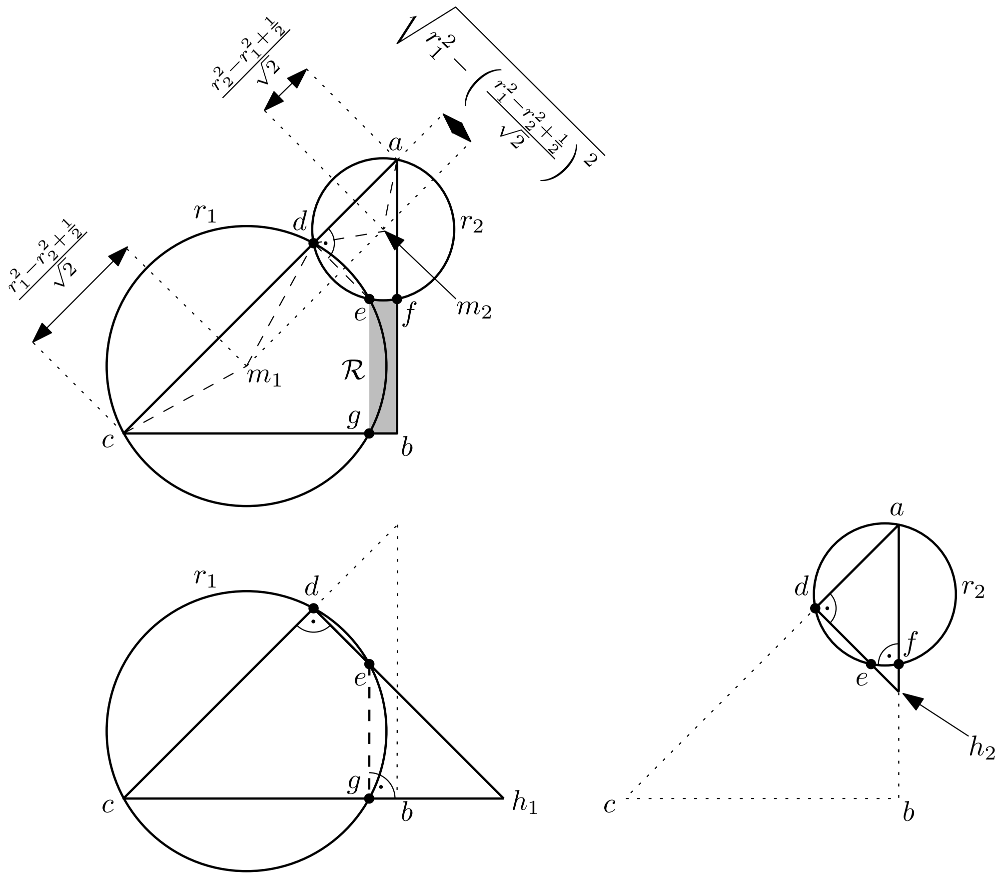

Figure 13 Top: Recursion Case 3. Bottom left: e, g have the same x-coordinates because e, g, h_1 are the corners of a halfsquare. Bottom right: e, f have the same y-coordinates because e, f, h_2 are the corners of a halfsquare.

the disks placed as depicted in Figure 13 top and the rectangle $\mathcal{R}$ cover the entire halfsquare, and that the remaining disks are sufficient to cover $\mathcal{R}$ using our rectangle covering algorithm.

▶ **Lemma 17.** *The union of $r_1, r_2, \mathcal{R}$ covers the entire halfsquare.*

Proof. In order to prove the lemma, we show four statements: (P1) e has the same x-coordinate as the intersection point g between bc and the boundary of r_1. (P2) e has the same y-coordinate as the intersection point f between ba and the boundary of r_2. (P3) The boundaries of r_1, r_2 intersect on ac. (P4) c, a lie on the boundaries of r_1, r_2 implying that the entire hypotenuse is covered by r_1 and r_2.

The combination of the four properties (P1)-(P4) will imply that $r_1, r_2, \mathcal{R}$ cover the entire halfsquare.

(P1) The segment de lies orthogonal to ac because m_1 and m_2 have the same distance to ac, see Figure 13 top. Let h_1 be the intersection point of the lines induced by bc and de, see Figure 13 bottom left. Thus, the triangle induced by c, d, h_1 is a halfsquare at which r_1 holds c, d on its boundary. Lemma 8 implies that the triangle induced by e, g, h_1 is a halfsquare. Hence, e, g have the same x-coordinate.

(P2) A symmetric approach implies that e, f have the same y-coordinate, see Figure 13 right.

(P3) The distance between c and the intersection point d_1 between ac and the boundary of r_1 is $2 \cdot \frac{r_1^2 - r_2^2 + 1/2}{\sqrt{2}}$. Analogously, the distance between a and the intersection point d_2 between ac and the boundary of r_2 is $2 \cdot \frac{r_2^2 - r_1^2 + 1/2}{\sqrt{2}}$. Thus, we obtain $|cd_1| + |ad_2| = \sqrt{2}$ implying $d_1 = d_2$, i.e., the boundaries of r_1 and r_2 intersect on ac.

(P4) The Pythagorean Theorem implies

$$|cm_1| = \sqrt{\left(\frac{r_1^2 - r_2^2 + \frac{1}{2}}{\sqrt{2}}\right)^2 + r_1^2 - \left(\frac{r_1^2 - r_2^2 + \frac{1}{2}}{\sqrt{2}}\right)^2} = r_1.$$

This means that c lies on the boundary of r_1.

A similar approach implies that a lies on the boundary of r_2. This concludes the proof. ◀

Let λ be the skew of $\mathcal{R}$. Assume that the two largest disks have radii $r_1 = \frac{1}{2} - \varepsilon_1$ and $r_2 = \frac{1}{2} - \varepsilon_2$ with $\varepsilon_1 \geq \frac{3}{116}$ or $\varepsilon_2 \geq \frac{3}{116}$. Using this assumption, we apply an interval arithmetic based program to establish that the remaining weight $\frac{1}{2} - r_1^2 - r_2^2$ is at least $\frac{195\lambda}{256}$ if $\lambda \leq \overline{\lambda}$ and at least $\frac{\lambda^2+2}{4}$ if $\lambda \geq \overline{\lambda}$. Hence, Theorem 4 implies that the remaining weight permits a covering of $\mathcal{R}$ if $\varepsilon_1 \geq \frac{3}{116}$ or $\varepsilon_2 \geq \frac{3}{116}$.

If $\varepsilon_1, \varepsilon_2 \leq \frac{3}{116}$, i.e., r_1, r_2 are approaching $\frac{1}{2}$, interval arithmetic fails because D is approaching the worst case instance made up of two disks $r_1 = r_2 = \frac{1}{2}$. Thus for the case $0 \leq \varepsilon_2 \leq \varepsilon_1 \leq \frac{3}{116}$, we give an analytic proof. In order to do this, we give upper bounds $8\varepsilon_2, 5\varepsilon_2$ for the dimensions of $\mathcal{R}$, see Lemmas 18 and 19[2]. Finally, we show by Lemma 20 that the remaining weight permits a covering of an rectangle with dimensions $8\varepsilon_2, 5\varepsilon_2$. This will imply that the remaining weight permits a covering of $\mathcal{R}$.

▶ **Lemma 18.** *If $0 \leq \varepsilon_2 \leq \varepsilon_1 \leq \frac{3}{116}$, the width $|bg|$ of $\mathcal{R}$ is upper-bounded by $8\varepsilon_2$.*

Proof. By construction of the midpoint m_1 of r_1, we have $\cos(\angle(d, c, m_1)) = \frac{r_1^2 - r_2^2 + 1/2}{\sqrt{2} r_1}$, see Figures 13 and 14.

The cosine addition theorem implies

$$\cos(\angle(m_1, c, g)) + \sin(\angle(m_1, c, g)) = \frac{r_1^2 - r_2^2 + \frac{1}{2}}{r_1}.$$

[2] Our upper bound $5\varepsilon_2$ on the height of $\mathcal{R}$ is smaller than our upper bound $8\varepsilon_2$ on the width of $\mathcal{R}$. Note, that this is a result of the approaches of Lemmas 18 and 19 while the rectangle $\mathcal{R}$ has larger height than width.

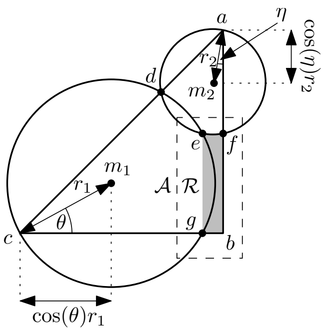

Figure 14 In Case (3), the width and height of the remaining rectangle $\mathcal{R}$ are equal to $1-\cos(\theta)r_1$ and $1-\cos(\eta)r_2$.

because $\angle(d,c,m_1) = \frac{\pi}{4} - \angle(m_1,c,g)$. For simplification, we set $\theta := \angle(m_1,c,g)$ and $c_1 := \frac{r_1^2-r_2^2+\frac{1}{2}}{r_1}$. This yields

$$
\begin{aligned}
& \sin(\theta) &&= c_1 - \cos(\theta)\\
\Leftrightarrow\quad & \sqrt{1-\cos^2(\theta)} &&= c_1 - \cos(\theta)\\
\Rightarrow\quad & 1-\cos^2(\theta) &&= (c_1-\cos(\theta))^2\\
\Leftrightarrow\quad & 1-\cos^2(\theta) &&= c_1^2 - 2c_1\cos(\theta) + \cos^2(\theta)\\
\Leftrightarrow\quad & \cos^2(\theta) &&= -\frac{1}{2}c_1^2 + c_1\cos(\theta) + \frac{1}{2}\\
\Leftrightarrow\quad & \cos^2(\theta) - c_1\cos(\theta) + \frac{1}{4}c_1^2 &&= -\frac{1}{4}c_1^2 + \frac{1}{2}\\
\Leftrightarrow\quad & \left(\cos(\theta) - \frac{1}{2}c_1\right)^2 &&= \frac{\left(2-c_1^2\right)}{4}\\
\Rightarrow\quad & \cos(\theta) - \frac{1}{2}c_1 &&= \frac{\sqrt{2-c_1^2}}{2}\\
\Leftrightarrow\quad & \cos(\theta) &&= \frac{c_1+\sqrt{2-c_1^2}}{2}.
\end{aligned}
$$

As $\cos(\theta)+\sin(\theta) = c_1$ with $\theta \in [0,\pi]$, we have $c_1 \in [1,\sqrt{2}]$. The first derivative of $\frac{c_1+\sqrt{2-c_1^2}}{2}$ is $\frac{1}{2}\left(1-\frac{c_1}{\sqrt{2-c_1^2}}\right)$ which is negative for $c_1 \in [0,\pi]$. This implies that $\frac{c_1+\sqrt{2-c_1^2}}{2}$ is monotonically decreasing in c_1. We have

$$\begin{aligned}
c_1 &= \frac{r_1^2 - r_2^2 + \frac{1}{2}}{r_1} \\
&\leq \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - \varepsilon_2\right)^2 + \frac{1}{2}}{\frac{1}{2} - \varepsilon_1} \\
&\overset{\varepsilon_2 \leq \frac{3}{116}}{\leq} (1 + 3\varepsilon_1)\left(\frac{1}{2} + \varepsilon_2 - \varepsilon_2^2\right) \\
&\leq (1 + 3\varepsilon_1)\left(\frac{1}{2} + \varepsilon_2\right) \\
&= 1 + 2\varepsilon_2 + 3\varepsilon_1 (1 + 2\varepsilon_2) \\
&= 1 + \underbrace{6\varepsilon_2}_{=:f_1}.
\end{aligned}$$

By applying that $\sqrt{1+\delta} \geq 1 + \delta$ holds for $\delta \in [-1, 0]$ ($\star$) and that $f_1^2 \leq f_1$ holds because $f_1 \leq 1$ (†), we lower bound $\cos(\theta)$ as follows:

$$\begin{aligned}
\cos(\theta) &= \frac{c_1 + \sqrt{2 - c_1^2}}{2} \\
&\overset{(\star)}{\geq} \frac{1 + f_1 + \sqrt{1 - f_1^2 - 2f_1}}{2} \\
&\overset{(\dagger)}{\geq} \frac{1 + f_1 + -f_1^2 - 2f_1}{2} \\
&\geq 1 - f_1.
\end{aligned}$$

Thus, $|bg| = 1 - 2\cos(\theta) r_1 \leq 1 - 2(1 - f_1)(\frac{1}{2} - \varepsilon_1) \leq 2\varepsilon_1 + f_1 = 8\varepsilon_2$, concluding the proof. ◀

The proof of Lemma 19 is symmetric to the proof of Lemma 18.

▶ **Lemma 19.** *If $0 \leq \varepsilon_2 \leq \varepsilon_1 \leq \frac{3}{116}$, the height $|bf|$ of $\mathcal{R}$ is upper-bounded by $5\varepsilon_2$.*

Proof. By construction of the midpoint m_2 of r_2, we have $\cos\left(\angle(d, a, m_2)\right) = \frac{r_2^2 - r_1^2 + 1/2}{\sqrt{2} r_2}$, see Figures 13 and 14. The cosine addition theorem implies

$$\cos\left(\angle(m_2, a, f)\right) + \sin\left(\angle(m_2, a, f)\right) = \frac{r_2^2 - r_1^2 + \frac{1}{2}}{r_2}.$$

because $\angle(d, a, m_2) = \frac{\pi}{4} - \angle(m_2, a, f)$. For simplification, we set $\eta := \angle(m_2, a, f)$ and

$c_2 := \frac{r_2^2 - r_1^2 + \frac{1}{2}}{r_2}$. This yields

$$\begin{aligned}
& \sin(\eta) &&= c_2 - \cos(\eta) \\
\Leftrightarrow\ & \sqrt{1-\cos^2(\eta)} &&= c_2 - \cos(\eta) \\
\Rightarrow\ & 1-\cos^2(\eta) &&= (c_2 - \cos(\eta))^2 \\
\Leftrightarrow\ & 1-\cos^2(\eta) &&= c_2^2 - 2c_2\cos(\eta) + \cos^2(\eta) \\
\Leftrightarrow\ & \cos^2(\eta) &&= -\frac{1}{2}c_2^2 + c_2\cos(\eta) + \frac{1}{2} \\
\Leftrightarrow\ & \cos^2(\eta) - c_2\cos(\eta) + \frac{1}{4}c_2^2 &&= -\frac{1}{4}c_2^2 + \frac{1}{2} \\
\Leftrightarrow\ & \left(\cos(\eta) - \frac{1}{2}c_2\right)^2 &&= \frac{\left(2-c_2^2\right)}{4} \\
\Leftrightarrow\ & \cos(\eta) - \frac{1}{2}c_2 &&= \frac{\sqrt{2-c_2^2}}{2} \\
\Leftrightarrow\ & \cos(\eta) &&= \frac{c_2 + \sqrt{2-c_2^2}}{2}.
\end{aligned}$$

As $\cos(\eta) + \sin(\eta) = c_2$ with $\eta \in [0, \pi]$, we have $c_2 \in [1, \sqrt{2}]$. The first derivative of $\frac{c_2+\sqrt{2-c_2^2}}{2}$ is $\frac{1}{2}\left(1 - \frac{c_2}{\sqrt{2-c_2^2}}\right)$ which is negative for $c_2 \in [0, \pi]$. This implies that $\frac{c_2+\sqrt{2-c_1^2}}{2}$ is monotonically decreasing in c_2. We have

$$\begin{aligned}
c_2 &= \frac{r_2^2 - r_1^2 + \frac{1}{2}}{r_2} \\
&\leq \frac{r_1^2 - r_1^2 + \frac{1}{2}}{r_2} \\
&\leq \frac{1}{2r_2} \\
&\leq 1 + \underbrace{3\varepsilon_2} =: f_2.
\end{aligned}$$

By applying that $\sqrt{1+\delta} \geq 1 + \delta$ holds for $\delta \in [-1, 0]$ $(\star)$ and that $f_2^2 \leq f_2$ holds because $f_2 \leq 1$ $(\dagger)$, we lower bound $\cos(\eta)$ as follows:

$$\begin{aligned}
\cos(\theta) &= \frac{c_2 + \sqrt{2-c_2^2}}{2} \\
&\overset{(\star)}{\geq} \frac{1 + f_2 + \sqrt{1 - f_2^2 - 2f_2}}{2} \\
&\overset{(\dagger)}{\geq} \frac{1 + f_2 + -f_2^2 - 2f_2}{2} \\
&\geq 1 - f_2.
\end{aligned}$$

Thus, $|bg| = 1 - 2\cos(\eta)r_2 \leq 1 - 2(1-f_2)(\frac{1}{2} - \varepsilon_2) \leq 2\varepsilon_2 + f_2 \leq 5\varepsilon_2$ as required. ◀

▶ **Lemma 20.** *If $0 \leq \varepsilon_2 \leq \varepsilon_1 \leq \frac{3}{116}$, the remaining weight $\frac{1}{2} - (\frac{1}{2} - \varepsilon_1)^2 - (\frac{1}{2} - \varepsilon_2)^2$ permits a covering of an $8\varepsilon_2 \times 5\varepsilon_2$ rectangle $\mathcal{A}$.*

Proof. The remaining weight is $\frac{1}{2} - \frac{1}{4} + \varepsilon_1 - \varepsilon_1^2 - \frac{1}{4} + \varepsilon_2 - \varepsilon_2^2 > \varepsilon_2 - 2\varepsilon_2^2 \geq \frac{110}{116}\varepsilon_2 \geq \frac{975}{32} \cdot \frac{3}{116}\varepsilon_2 \geq \frac{195}{256} \cdot 8\varepsilon_2 \cdot 5\varepsilon_2$. Thus, Theorem 4 implies that the remaining weight permits a covering of $\mathcal{A}$ which in turn covers $\mathcal{R}$. ◀

4.3.6 Details for Case (4)

In Case (4), we prove that the disks that remain after placing r_1, r_2 as depicted in Figure 8(c) suffice to cover $\mathcal{R}$.

▶ **Lemma 21.** *The remaining weight $\frac{1}{2} - r_1^2 - r_2^2$ permits a covering of $\mathcal{R}$.*

Proof. Consider the inequality

$$-(r_1+r_2)^2 - \frac{5}{12} + \frac{11\sqrt{2}}{12}(r_1+r_2) \geq -\frac{r_1 r_2}{6} \tag{13}$$

restricted to $r_1 + r_2 \in \left[\frac{5}{6\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. The term on the left hand side achieves its minimum 0 at $r_1 + r_2 = \frac{1}{\sqrt{2}}$ and $r_1 + r_2 = \frac{5}{6\sqrt{2}}$. Hence, the left hand side is non-negative while the right hand side is non-positive ensuring that Inequality 13 is satisfied.

Lemma 8 implies that the triangles covered by r_1, r_2 are halfsquares. Thus, the lengths of the sides of $\mathcal{R}$ lie between 1 and $\frac{1}{2}$ because $1 - \sqrt{2}r_1 \geq 1 - \sqrt{2}\frac{1}{2\sqrt{2}}$ as $r_1 \leq \frac{1}{2\sqrt{2}}$. Thus, $\mathcal{R}$ has a skew $\lambda := \frac{1-\sqrt{2}r_2}{1-\sqrt{2}r_1}$ no larger than 2.

Inequality 13 is equivalent to

$$\frac{1}{2} - r_1^2 - r_2^2 \geq \frac{11}{12}\left(1 - \sqrt{2}r_1\right)\left(1 - \sqrt{2}r_2\right)$$

which implies that the remaining weight is at least $\frac{11}{12}|\mathcal{R}| \geq \frac{195}{256}|\mathcal{R}|$. Hence, Theorem 4 implies that the remaining weight permits a covering of $\mathcal{R}$. ◀

4.3.7 Details for Case (5)

Recall that, in Case (5) (see Figure 8(d)), we have $r_1 < {}^1\!/_{2\sqrt{2}}$ and $r_1 + r_2 < {}^5\!/_{6\sqrt{2}}$. Thus, $r_1 \geq r_2$ implies $r_2 \leq {}^5\!/_{12\sqrt{2}}$. Furthermore, recall that we build a set D_s of disks of non-increasing weight, starting with r_2. By construction, this set has total weight $w(D_s)$ at least $w(D_s) \geq \frac{195 r_2}{256\sqrt{\hat{\sigma}}}$. Because the largest disk in D_s has weight r_2^2, the total weight of D_s is at most $w(D_s) \leq \frac{195 r_2}{256\sqrt{\hat{\sigma}}} + r_2^2 < {}^1\!/_2 - \frac{1}{(2\sqrt{2})^2} < {}^1\!/_2 - r_1^2$. This implies that we always have enough weight in disks $r_2, r_3, \ldots$ to ensure that we will eventually exceed the required weight. Furthermore, recall that we use D_s to cover a rectangular strip $\mathcal{R}$ of width 1 and height $s = {}^{256 w(D_s)}\!/_{195}$. Using the bounds on D_s, we can bound this height by $\frac{r_2}{\sqrt{\hat{\sigma}}} \leq s < \frac{r_2}{\sqrt{\hat{\sigma}}} + \frac{256 r_2^2}{195} < 1$.

We begin by showing that this covering is actually guaranteed to succeed using Lemma 5. We note that for $\sigma = \hat{\sigma} = {}^{195\sqrt{5257}}\!/_{16384}$, the covering density $E(\sigma)$ guaranteed by Lemma 5 is ${}^{195}\!/_{256}$ disk weight per unit of rectangle area. By construction, the area of $\mathcal{R}$ is $s = {}^{256 w(D_s)}\!/_{195} = {}^{w(D_s)}\!/_{E(\hat{\sigma})}$. Therefore, a successful covering is guaranteed by Lemma 5 if the largest disk r_2 in D_s satisfies the size bound condition of the lemma for $\sigma = \hat{\sigma}$. Note that the situation in the lemma is normalized such that the shorter side of the rectangle has length 1. To check the size bound condition of the Lemma 5, we thus scale the rectangle and the disks by a factor of ${}^1\!/_s$. We hence need to verify that $\hat{\sigma} \geq \frac{r_2^2}{s^2}$, which follows from the lower bound on s and thus from the construction of D_s. Thus, we now know that covering $\mathcal{R}$ is possible with D_s.

It only remains to prove that the remaining disks, which include r_1, always suffice to cover the remaining halfsquare $\mathcal{H}$ recursively. For this, we consider the area $\Delta_s = s - {}^{s^2}\!/_2$ of the original halfsquare that we cover using D_s, i.e., the area s of $\mathcal{R}$ minus the part of $\mathcal{R}$ that is not in our original halfsquare. We are given a total disk weight of ${}^1\!/_2$ to cover a halfsquare of area ${}^1\!/_2$; thus, on average, we have to cover one unit of halfsquare area for each unit of disk weight. In order to prove that the remaining disks suffice to cover $\mathcal{H}$, we thus prove that the halfsquare area Δ_s covered by D_s satisfies $E_\Delta := {}^{w(D_s)}/_{\Delta_s} \leq 1$, i.e., that we cover at least one unit of halfsquare area for each unit of weight in D_s. We have

$$E_\Delta = \frac{\frac{195s}{256}}{s(1-\frac{s}{2})} = \frac{195}{128(2-s)} \leq 1 \Leftrightarrow s \leq \frac{61}{128}.$$

Using our upper bound on s and solving the resulting quadratic inequality for $r_2 > 0$, we get

$$\frac{r_2}{\sqrt{\hat{\sigma}}} + \frac{256r_2^2}{195} \leq \frac{61}{128} \Leftrightarrow r_2 \leq \frac{1}{128}\sqrt{\frac{(195(320677 + 2048\sqrt{5257}))}{10514}} - \frac{\sqrt{195}}{4\sqrt[4]{5257}} \approx 0.318775,$$

which follows from the fact that we have $r_2 \leq {}^5/_{12\sqrt{2}} \approx 0.29463$.

Therefore, partially covering the halfsquare with D_s is actually more efficient than necessary; therefore, after using the disks in D_s, enough weight remains to cover $\mathcal{H}$ recursively.

4.4 Interval arithmetic: implementation details

Implementing interval arithmetic operations is a task that involves several challenges. For floating-point numbers with hardware support, such as the standard `double` type in C/C++ programs, which on usual computers is implemented using a 64-bit IEEE 754 floating-point representation, for performance reasons, one generally wants to use the operations directly supported by the underlying hardware, if those are available and correctly rounded. Usually, the correctly rounded operations include $a + b, a - b, a \cdot b, a/b, 1/a$ and $\sqrt{a}$ together with non-rounding (exact) operations such as $-a, \min(a, b), \max(a, b)$ and $|a|$. For other operations such as $\sin, \cos, \tan$ or similar, one has to rely on correctly-rounded software implementations such as those provided by MPFR; these are often orders of magnitude slower than their hardware-supported counterparts.

On CPUs, the rounding mode is usually controlled via a set of global (or rather, thread-local) flags; typically, the rounding modes *round to nearest ties to even*, *round up*, *round down* and *round towards zero* are available, with round to nearest being the default.

Aside from their use in interval arithmetic, non-default rounding modes such as *round down* are rarely used. Therefore, hardware is not heavily optimized w.r.t. code that changes rounding mode often; changing the rounding mode can thus be a rather expensive operation. Thus, most interval arithmetic implementations, such as the one provided by CGAL, try to avoid frequent rounding mode changes. For most operations, this can easily be achieved by observing that $-(-a - b)$ computed in round-down mode is equivalent to computing $a + b$ in round-up mode, or similar observations for other operations. This is known as the *sign-switching trick* and allows one to perform interval arithmetic additions, subtractions, multiplications and divisions without touching the rounding mode, so long as one initially sets either round down or round up. The only hardware-supported instruction with correct rounding that actually requires changes to the rounding mode flag is thus $\sqrt{a}$.

Another, more problematic challenge is the following. Owing to the infrequent and specialist use of non-default rounding modes and the design decision to make different rounding modes accessible via a thread-local flag instead of instruction prefixes (which are common in GPGPU floating-point implementations and much nicer to use in programs), C/C++ *compiler support* for different rounding modes is rather poor. In this context, *compiler support* means that the compiler can be made aware that the program may change its rounding mode away from the default. This is important, as compilers typically assume this will not happen, which causes them to silently break the program they are compiling by *optimizations* that only preserve the result in the default rounding mode. For instance, a non-supporting compiler will usually treat $a + b$ and $-(-a - b)$ as duplicate computations of the same value and remove one of the two.

Some major compilers, such as clang, do not officially support the use of non-default rounding modes at all. While GCC and MSVC both claim support for rounding-mode changes given appropriate flags at compile time, we encountered several situations where they still broke programs using an outdated version of CGAL's interval arithmetic `sqrt`: by manually analyzing the generated assembly, we found that in some cases, depending on where the function was inlined, they reordered rounding mode changes around square root operations such that both square root operations in an interval `sqrt` were performed under the same rounding mode.

Therefore, in many cases, using different rounding modes actually requires some hacky tricks to work around non-supporting compilers or compiler bugs. In CGAL's implementation, this is done by *opacifying* the values of inputs to rounding operations. In the process, the inputs are either put as input/output operands into an (empty) inline assembly block or written to and read back from a `volatile` variable. Unless the compiler is clever enough to analyze the contents of the inline assembly, in either case, its optimizer has to assume that the values are changed arbitrarily; this typically prevents any unwanted optimizations from occurring. However, it did not appear to always prevent reordering issues; moreover, it can lead to severely suboptimal assembly being generated. Using CGAL's implementation, in some cases, we observed tens of superfluous instructions in a row, all writing the same value to the same memory address without ever reading it back. In addition, it may happen that, at some point, a compiler vendor decides to analyze inline assembly, at which point parts of the opacifying technique may cease to work. Furthermore, there is some genuine potential for compiler optimizations: for instance, if the result of one interval addition is directly consumed by another interval addition, e.g., while computing $a + b + c$, the sign-switching trick first flips the sign of one of the interval bounds only to flip it again before performing the second addition; these duplicate switches can safely be eliminated.

These reasons, together with the fact that we needed interval arithmetic operations such as $\sin, \cos$ and $\tan$ not offered by CGAL, led us to implement our own interval arithmetic type based on 64-bit IEEE 754 floating-point values. Each interval is stored as two double-precision floating point numbers packed in a 128-bit vector. Instead of CGAL's opacification approach, we implement all non-rounding operations (such as sign-switches, absolute values, min and max) using standard operators or compiler intrisics, which allow the compiler to perform regular optimizations on them, such as eliminating duplicate sign-switches. All rounding operations (additions, subtractions, multiplications, divisions and square roots) are performed in inline assembly to prevent wrong optimizations; for other operations, we rely on the MPFR library to provide correctly-rounded implementations. This has the added benefit of avoiding any ambiguity as to which floating-point implementation is actually used; we cannot, for instance, run into issues of double rounding that could occur if the compiler were to select the x87 FPU to implement certain operations for some reason. We also ensure to set the floating-point environment up so that other potentially problematic options, such as flushing denormal values to zero before or after operations, are deselected; in our prover, except for square root operations where we switch to round up temporarily, the rounding mode is only changed to round down once at program start.

This approach is feasible for us, as we do not need to provide the same support for a vast variety of hardware architectures and compilers that CGAL, or other interval arithmetic libraries, have to offer.

5 Conclusion

We have presented worst-case optimal algorithms for covering equilateral triangles and both obtuse isosceles triangles and isosceles triangles with sufficiently small apex angle. This gives rise to numerous followup questions and extensions, for instance how to extend our results to 3D, to all isosceles triangles, or to other shapes such as trapezoids or convex shapes. Similar to optimal packings of disks, computing optimal coverings by disks is quite difficult. However, while the complexity of deciding whether a given set of disks can be packed into a unit square is known to be NP-hard [13], it is still open whether it is NP-hard to decide whether a given set of disks can be used to cover a triangle. Ironically, it is the higher practical difficulty of covering by disks that makes it challenging to apply the same proof idea in a straightforward manner.

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