A SAT Attack on Erdős–Szekeres Numbers in $\mathbb{R}^d$ and the Empty Hexagon Theorem

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Abstract

A famous result by Erdős and Szekeres (1935) asserts that, for all $k, d \in \mathbb{N}$, there is a smallest integer $n = g^{(d)}(k)$ such that every set of at least n points in $\mathbb{R}^d$ in general position contains a k-*gon*, that is, a subset of k points in convex position. We present a SAT model based on acyclic chirotopes (oriented matroids) to investigate Erdős–Szekeres numbers in small dimensions. SAT instances are solved using modern SAT solvers and unsatisfiability results are verified using DRAT certificates. We show $g^{(3)}(7) = 13$, $g^{(4)}(8) \leq 13$, and $g^{(5)}(9) \leq 13$, which are the first improvements for decades. For the setting of k-*holes* (i.e., k-gons with no other points in the convex hull), where $h^{(d)}(k)$ denotes the smallest integer n such that every set of at least n points in $\mathbb{R}^d$ in general position contains a k-hole, we show $h^{(3)}(7) \leq 14$, $h^{(4)}(8) \leq 13$, and $h^{(5)}(9) \leq 13$. All obtained bounds are sharp in the setting of acyclic chirotopes and we conjecture them to be sharp also in the original setting of point sets. Last but not least, we verify all previously known bounds and, in particular, we present the first computer-assisted proof for the existence of 6-holes in sufficiently planar point sets by verifying Gerken's estimate $h^{(2)}(6) \leq g^{(2)}(9)$.

Keywords and phrases Erdős–Szekeres theorem, empty hexagon theorem, higher dimensional point set, acyclic chirotope, oriented matroid, k-gon, k-hole, Boolean satisfiability (SAT), computer-assisted proof, automated reasoning

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Supplementary Material Source code and data is available online [39].

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1 Introduction

A set of points S in the plane is *in general position* if no three points of S lie on a common line and *in convex position* if all point of S lie on the boundary of the convex hull conv(S). The classical Erdős–Szekeres Theorem [14] asserts that every sufficiently large point set in the plane in general position contains a k-*gon*, that is, a subset of k points in convex position.

▶ **Theorem 1** ([14], The Erdős–Szekeres Theorem). *For every integer $k \geq 3$, there is a smallest integer $n = g^{(2)}(k)$ such that every set of at least n points in general position in the plane contains a k-gon.*

Erdős and Szekeres showed that $g^{(2)}(k) \leq \binom{2k-4}{k-2} + 1$ [14] and constructed point sets of size 2^{k-2} without k-gons [15], which they conjectured to be extremal. There were several improvements of the upper bound in the past decades, each of magnitude $4^{k-o(k)}$, and in 2016, Suk showed $g^{(2)}(k) \leq 2^{k+o(k)}$ [43]. Shortly after, Holmsen et al. [25] slightly improved the error term in the exponent and showed that the bound also applies in the more general setting of acyclic chirotopes (see Section 3 for the definition). The lower bound $g^{(2)}(k) \geq 2^{k-2} + 1$ is known to be sharp for $k \leq 6$. The value $g^{(2)}(4) = 5$ was determined by Klein in 1933 and

$g^{(2)}(5) = 9$ was soon later determined by Makai; for a proof see [27], [10] or [36]. However, several decades passed until Szekeres and Peters [44] managed to show $g^{(2)}(6) = 17$ by using computer assistance. While their computer program uses thousands of CPU hours, we have developed a SAT framework [41] which allows to verify this result within only 2 CPU hours, and an independent verification of their result using SAT solvers was done by Marić [34].

1.1 Planar k-holes

In the 1970's, Erdős [16] asked whether every sufficiently large point set contains a *k-hole*, that is, a k-gon with the additional property that no other point lies in its convex hull. In the same vein as $g^{(2)}(k)$, we denote by $h^{(2)}(k)$ the smallest integer such that every set of at least $h^{(2)}(k)$ points in general position in the plane contains a k-hole. This variant differs significantly from the original setting as Horton [26] managed to construct arbitrarily large point sets without 7-holes (see [46] for a generalization called *Horton sets*). While Harborth [24] showed $h^{(2)}(5) = 10$, the existence of 6-holes remained open until 2006, when Gerken [23] and Nicolás [37] independently showed that sufficiently large point sets contain 6-holes. We also refer the interested reader to Valtr's simplified proof [48]. Today the best bounds concerning 6-holes are $30 \leq h^{(2)}(6) \leq 1717$. The lower bound is witnessed by a set of 29 points without 6-holes that was found via a simulated annealing approach by Overmars [38]. The currently best upper bound $h^{(2)}(6) \leq 1717$ is by Gerken[1], who made an elaborate case distinction on about 30 pages to show that every 9-gon – independent of how many interior points it contains – yields a 6-hole. Using the estimate $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ by Tóth and Valtr [45], he then concluded $h^{(2)}(6) \leq g^{(2)}(9) \leq \binom{11}{5} + 1 = 1717$. The estimates by Nicolás [37] and Valtr [48] are weaker.

1.2 Higher dimensions

The notions *general position* (no $d+1$ points in a common hyperplane), *k-gon* (a set of k points in convex position), and *k-hole* (a k-gon with no other points in the convex hull) naturally generalize to higher dimensions, and so does the Erdős–Szekeres Theorem [14, 13] (cf. [36]). We denote by $g^{(d)}(k)$ and $h^{(d)}(k)$ the minimum number of points in $\mathbb{R}^d$ in general position that guarantee the existence of k-gon and k-hole, respectively. In contrast to the planar case, the asymptotic behavior of the higher dimensional Erdős–Szekeres numbers $g^{(d)}(k)$ remains unknown for dimension $d \geq 3$. While a dimension-reduction argument by Károlyi [28] combined with Suk's bound [43] shows

$$g^{(d)}(k) \leq g^{(d-1)}(k-1) + 1 \leq \ldots \leq g^{(2)}(k-d+2) + d - 2 \leq 2^{(k-d)+o(k-d)}$$

for $k \geq d \geq 3$, the currently best asymptotic lower bound $g^{(d)}(k) = \Omega(c^{\sqrt[d-1]{k}})$ with $c = c(d) > 1$ is witnessed by a construction by Károlyi and Valtr [29]. For dimension 3, Füredi conjectured $g^{(3)}(k) = c^{\Theta(\sqrt{k})}$ (unpublished, cf. [35, Chapter 3.1]). Very recently, Pohoata and Zakharov [50] presented a subexponential bound for all dimension $d \geq 3$.

[1] Koshelev [32] claimed that $h^{(2)}(6) \leq 463$, but his proof is incomplete. In his 50 pages, written in Russian (see [31] for a 3-page extended abstract written in English), he classifies how potential counterexamples may look like and then the article suddenly stops. In fact, in the last sentence of Section 2, Koshelev even writes that the completion of the proof will be given in the next publication, which in the author's knowledge did not appear.

	$k=4$	5	6	7	8	9	10	11
$d=2$	5	9	17					
3	4	6	9	13*				
4	4	5	7	9	≤ 13*			
5	4	5	6	8	10	≤ 13*		
6	4	5	6	7	9	11	13	

Table 1 Known values and bounds for $g^{(d)}(k)$. Entries marked with a star (*) are new. Entries left blank are upper-bounded by the estimate $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ [45] and the dimension-reduction argument [28].

1.3 Higher dimensional holes

Since Valtr [47] gave a construction for any dimension d without $d^{d+o(d)}$-holes, generalizing the idea of Horton [26], the central open problem about higher dimensional holes is to determine the largest value $k = H(d)$ such that every sufficiently large set in d-space contains a k-hole. Note that with this notation we have $H(2) = 6$ because $h^{(2)}(6) < \infty$ [23, 37] and $h^{(2)}(7) = \infty$ [26]. Very recently Bukh, Chao and Holzman [11] presented a construction without 2^{7d}-holes, which further improves Valtr's bound and shows $H(d) < 2^{7d}$. Another remarkable result is by Conlon and Lim [12], who recent generalized *squared Horton sets* [46] to higher dimensions (point sets which are slight perturbations of d-dimensional grids and which do not contain large holes).

On the other hand, the dimension-reduction argument by Károlyi [28] also applies to k-holes, and therefore

$$h^{(d)}(k) \leq h^{(d-1)}(k-1) + 1 \leq \ldots \leq h^{(2)}(k-d+2) + d - 2.$$

This inequality together with $h^{(2)}(6) < \infty$ implies that $h^{(d)}(d+4) < \infty$ and hence $H(d) \geq d+4$. A slightly better bound of $H(d) \geq 2d+1$ was provided by Valtr [47], who showed $h^{(d)}(2d+1) \leq g^{(d)}(4d+1)$. However, already in dimension 3 the gap between the upper and the lower bound of $H(3)$ remains huge: while there are arbitrarily large sets without 23-holes [47], already the existence of 8-holes remains unknown ($7 \leq H(3) \leq 22$).

1.4 Precise values

As discussed before, for the planar k-gons $g^{(2)}(5) = 9$, $g^{(2)}(6) = 17$, $h^{(2)}(5) = 10$, and $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ are known. For planar k-holes, $h^{(2)}(5) = 9$, $30 \leq h^{(2)}(6) \leq 463$, and $h^{(2)}(k) = \infty$ for $k \geq 7$ are known.

While the values $g^{(d)}(k) = h^{(d)}(k) = k$ for $k \leq d+1$ and $g^{(d)}(d+2) = h^{(d)}(d+2) = d+3$ are easy to determine (cf. [7]), Bisztriczky et al. [7, 6, 36] showed $g^{(d)}(k) = h^{(d)}(k) = 2k - d - 1$ for $d + 2 \leq k \leq \frac{3d}{2} + 1$. This, in particular, determines the values for $(k, d) = (3, 5), (4, 6), (4, 7), (5, 7), (5, 8)$ and shows $H(d) \geq \lfloor \frac{3d}{2} \rfloor + 1$. For $k > \frac{3d}{2} + 1$ and $d \geq 3$, Bisztriczky and Soltan [7] moreover determined the values $g^{(3)}(6) = h^{(3)}(6) = 9$. Tables 1 and 2 summarize the currently best bounds for k-gons and k-holes in small dimensions.

2 Our results

In this article we generalize our SAT framework from [41] to higher dimensions (see Section 4). Our framework models k-gons and k-holes in terms of acyclic chirotopes, which are well-studied combinatorial structures generalizing point sets in a natural manner (see Section 3).

	$k=4$	5	6	7	8	9	10	11	12	13
$d=2$	5	10	30..463	∞	∞	∞	∞	∞	∞	∞
3	4	6	9	≤ 14*	?	?	?	?	?	?
4	4	5	7	9	≤ 13*	!	?	?	?	?
5	4	5	6	8	10	≤ 13*	!	!	?	?
6	4	5	6	7	9	11	13	!	!	!

Table 2 Known values and bounds for $h^{(d)}(k)$. Entries marked with a star (*) are new. Entries marked with an exclamation mark (!) are finite because of the estimate $h^{(d)}(2d+1) \leq g^{(d)}(4d+1)$ [47]. Entries marked with a question mark (?) are not known to be finite.

Using this framework, we have been able to verify previously known results and to prove the following new upper bounds for higher dimensional Erdős–Szekeres numbers and for the variant of k-holes in dimensions 3, 4 and 5, which we moreover conjecture to be sharp. Details about the computations are given in Section 5.

▶ **Theorem 2.** *$g^{(3)}(7) = 13$, $h^{(3)}(7) \leq 14$, $g^{(4)}(8) \leq h^{(4)}(8) \leq 13$, and $g^{(5)}(9) \leq h^{(5)}(9) \leq 13$.*

Our SAT framework also found chirotopes that witness that all bounds from Theorem 2 are sharp in the more general setting of acyclic chirotopes. All witnesses are available as supplemental data [39]. For the chirotope in rank 4 without 7-gons we managed to find a realization. The explicit coordinates of this set of 12 points in $\mathbb{R}^3$ without 7-gons are:

$$\begin{aligned}
\{ \quad & (526,446,232), & (0,756,64), && (612,660,342), && (708,638,193), \\
& (546,563,134), & (616,622,174), && (414,0,370), && (548,594,151), \\
& (884,1334,722), & (452,668,180), && (587,659,156), && (579,692,0) \quad \}.
\end{aligned}$$

For the other witnessing chirotopes we could not find a realization as point set. In fact, we classified all those examples as non-realizable using the method of bi-quadratic final polynomials by Bokowski and Richter [9] (see Chapter 9.7 in [40] for an outline). However, since the majority of chirotopes is known to be non-realizable, it is not surprising that our programs struggled with finding realizable examples. We conjecture that there are indeed realizable witnesses (even though they are hard to find) and that all bounds from Theorem 2 are sharp in the original setting.

Last but not least, in Section 5.2 we use our SAT framework to verify the previously known bounds $g^{(2)}(5) \leq 9$ (Makai 1935), $g^{(2)}(6) \leq 17$ [44], $h^{(2)}(5) \leq 10$ [24], and $h^{(3)}(6) \leq 9$ [7] and in Section 5.3 we present a computer-assisted proof for the existence of 6-holes in sufficiently large planar point sets. To our knowledge, this is the first alternative proof of Gerken's, which covers about 30 pages of case distinction [23].

▶ **Theorem 3** (The Empty Hexagon Theorem [23])**.** *$h^{(2)}(6) \leq g^{(2)}(9)$.*

All mentioned bounds apply to the more general setting of acyclic chirotopes because all computer-assisted computations were performed on chirotopes and Lemma 1 from [48][2] (which plays a central role in the proof of Theorem 3) applies to acyclic chirotopes of rank 3. Also the estimate $g^{(2)}(k) \leq \binom{2k-5}{k-2} + 1$ [45] which finally gives the currently best estimate $h^{(2)}(6) \leq g^{(2)}(9) \leq 1717$ applies to acyclic chirotopes of rank 3.

[2] In the proof only the combinatorial information of the point set is used, not the actual coordinates.

3 Preliminaries

Let $S = \{p_1, \ldots, p_n\}$ be a set of n labeled points in $\mathbb{R}^d$ in general position with coordinates $p_i = (x_{i,1}, \ldots, x_{i,d})$. We assign to each $(d+1)$-tuple $i_0, \ldots, i_d$ a sign to indicate whether the $d+1$ corresponding points $p_{i_0}, \ldots, p_{i_d}$ are positively or negatively oriented. Formally, we define a mapping $\chi_S : \{1, \ldots, n\}^d \to \{-1, 0, +1\}$ with

$$\chi_S(i_0, \ldots, i_d) = \operatorname{sgn}\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ p_{i_0} & p_{i_1} & \ldots & p_{i_d} \end{pmatrix} = \operatorname{sgn}\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_{i_0,1} & x_{i_1,1} & \ldots & x_{i_d,1} \\ \vdots & \vdots & & \vdots \\ x_{i_0,d} & x_{i_1,d} & \ldots & x_{i_d,d} \end{pmatrix}.$$

The properties of the mapping χ_S are captured in the following definition (cf. [8, Definition 3.5.3]).

▶ **Definition 4** (Chirotope). *A mapping $\chi\colon \{1, \ldots, n\}^r \to \{-1, 0, +1\}$ is a* chirotope *of rank r if the following three properties are fulfilled:*

(i) *χ is not identically zero;*

(ii) *for every permutation σ and indices $a_1, \ldots, a_r \in \{1, \ldots, n\}$,*

$$\chi(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = \operatorname{sgn}(\sigma) \cdot \chi(a_1, \ldots, a_r);$$

(iii) *for all indices $a_1, \ldots, a_r, b_1, \ldots, b_r \in \{1, \ldots, n\}$,*

$$\textit{if} \quad \chi(b_i, a_2, \ldots, a_r) \cdot \chi(b_1, \ldots, b_{i-1}, a_1, b_{i+1}, \ldots, b_r) \geq 0 \quad \textit{holds for all } i = 1, \ldots, r$$
$$\textit{then we have} \quad \chi(a_1, \ldots, a_r) \cdot \chi(b_1, \ldots, b_r) \geq 0.$$

It is not hard to verify that the mapping χ_S is a chirotope of rank $r = d+1$. The first item of Definition 4 is fulfilled because the point set S is in general position and therefore not all points lie in a common hyperplane. The second item is fulfilled because, by the properties of the determinant, we have

$$\det(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = \operatorname{sgn}(\sigma) \cdot \det(a_1, \ldots, a_r)$$

for any r-dimensional vectors $a_1, \ldots, a_r$ and any permutation σ of the indices $\{1, \ldots, r\}$. Since we can consider the homogeneous coordinates $(1, p_1), \ldots, (1, p_n)$ of our d-dimensional point set S as $(d+1)$-dimensional vectors, the above relation also has to be respected by χ_S. To see that χ_S fulfills the third item, recall that the well-known Graßmann–Plücker relations (see e.g. [8, Chapter 3.5]) assert that any r-dimensional vectors $a_1, \ldots, a_r, b_1, \ldots, b_r$ fulfill[3]

$$\det(a_1, \ldots, a_r) \cdot \det(b_1, \ldots, b_r) = \sum_{i=1}^{r} \det(b_i, a_2, \ldots, a_r) \cdot \det(b_1, \ldots, b_{i-1}, a_1, b_{i+1}, \ldots, b_r).$$

[3] The Graßmann–Plücker relations can be derived using Linear Algebra basics as outlined: Consider the vectors $a_1, \ldots, a_r, b_1, \ldots, b_r$ as an $r \times 2r$ matrix and apply row additions to obtain the echelon form. If the first r columns form a singular matrix, then $\det(a_1, \ldots, a_n) = 0$ and both sides of the equation vanish by a simple column multiplication argument. Otherwise, we can assume that the first r columns form an identify matrix. Since the determinant is invariant to row additions, none of the terms in the Graßmann–Plücker relations is effected during the transformation, and the statement then follows from Laplace expansion.

In particular, if all summands on the right-hand side are non-negative then also the left-hand side must be non-negative.

On the other hand, only few chirotopes come from an actual point set. Such chirotopes are called *realizable*. For each fixed dimension d, only $2^{\Theta(n \log n)}$ of the $2^{\Theta(n^d)}$ rank $d+1$ chirotopes are realizable by point sets in $\mathbb{R}^d$ and moreover the problem of deciding realizability is known to be ETR-complete where $\mathsf{NP} \subseteq \mathsf{ETR} \subseteq \mathsf{PSPACE}$ (cf. Chapters 7.4 and 8.7 in [8]).

In this article, we formulate Erdős–Szekeres-type problems in terms of chirotopes. In a general chirotope, we can invert the sign of all r-tuples which contain a fixed index i and again obtain a valid chirotope. However, if we apply such a reversal to a chirotope, which is induced by a point set, the obtained chirotope might not be induced by any point set. More specifically, if the points $p_1, \dots, p_r$ determine a positively oriented simplex (i.e., $\chi(1, 2, \dots, r) = +$), then another point p_{r+1} might lie inside the simplex (i.e., $\chi(1, 2, \dots, i-1, r+1, i+1, \dots, r) = +$ for all $i = 1, \dots, r$), but it cannot fulfill $\chi(1, 2, \dots, i-1, r+1, i+1, \dots, r) = -$ for all $i = 1, \dots, r$. Chirotopes with this property are called *acyclic* (cf. Chapter 3.4 and page 21 in [8]). It is worth noting that besides chirotopes there exist several cryptomorphic axiomatizations for oriented matroids and that acyclic chirotopes have been investigated under several different names: *abstract order types* (cf. [2, 33, 3, 20, 21]), *CC systems* (cf. [30]), or *pseudoconfigurations of points* (cf. [8]). We refer the interested reader to the homepage of oriented matroids [19]; for the non-degenerate rank 3 case, see [1]. It is also worth noting that for any acyclic chirotope of rank 3 there exists a reordering of the elements such that it becomes a *signotope* of rank 3 (cf. [18]).

In Section 4 we present the SAT encoding for acyclic chirotopes. While the chirotope axioms from Definition 4 require $\Theta(n^{2r})$ constraints, we can significantly reduce this number to $\Theta(n^{r+2})$ by using an axiom system based on the 3-term Graßmann–Plücker relations.

▶ **Theorem 5** (3-term Graßmann–Plücker relations, [8, Theorem 3.6.2]). *A sign mapping $\chi\colon \{1, \dots, n\}^r \to \{-1, 0, +1\}$ is a non-degenerate chirotope of rank r if the following three properties are fulfilled:*

(i) *for every r distinct indices $a_1, \dots, a_r \in \{1, \dots, n\}$,*

$$\chi(a_1, \dots, a_r) \neq 0;$$

(ii) *for every permutation σ and indices $a_1, \dots, a_r \in \{1, \dots, n\}$,*

$$\chi(a_{\sigma(1)}, \dots, a_{\sigma(r)}) = \operatorname{sgn}(\sigma) \cdot \chi(a_1, \dots, a_r);$$

(iii) *for any $a_1, \dots, a_r, b_1, b_2 \in \{1, \dots, n\}$,*

$$\begin{array}{ll} \textit{if} & \chi(b_1, a_2, \dots, a_r) \cdot \chi(a_1, b_2, a_3, \dots, a_r) \geq 0 \\ \textit{and} & \chi(b_2, a_2, \dots, a_r) \cdot \chi(b_1, a_1, a_3, \dots, a_r) \geq 0 \\ \textit{then} & \chi(a_1, a_2, \dots, a_r) \cdot \chi(b_1, b_2, a_3, \dots, a_r) \geq 0. \end{array}$$

3.1 Gons and holes

Carathéodory's theorem asserts that a d-dimensional point set is in convex position if and only if all $(d+2)$-element subsets are in convex position. Now that a point $p_{i_{d+1}}$ lies in the convex hull of $\{p_{i_0}, \dots, p_{i_d}\}$ if and only if $\chi(i_0, \dots, i_d) = \chi(i_0, \dots, i_{j-1}, i_{d+1}, i_{j+1}, \dots, i_d)$ holds for every $j \in \{0, \dots, d\}$, we can fully axiomize k-gons and k-holes solely using the information of the chirotope, that is, the relative position of the points. Note that $\chi(i_0, \dots, i_d) = \chi(i_0, \dots, i_{j-1}, i_{d+1}, i_{j+1}, \dots, i_d)$ holds if and only if the two points p_{i_j} and $p_{i_{d+1}}$ lie on the same side of the hyperplane determined by $\{p_{i_0} \dots, p_{i_{j-1}}, p_{i_{j+1}}, \dots, p_{i_d}\}$.

4 The SAT encoding

For the proof of Theorem 2, we proceed as follows: To show $g^{(d)}(k) \leq n$ or $h^{(d)}(k) \leq n$, respectively, assume towards a contradiction that there exists a set S of n points in $\mathbb{R}^d$ in general position, which does not contain any k-gon or k-hole, respectively.

The point set S induces an acyclic chirotope χ of rank $r = d+1$, which can be encoded using Boolean variables: For every r distinct indices $a_1, \ldots, a_r$, the variable $X_{a_1,\ldots,a_r}$ indicates whether $\chi(a_1, \ldots, a_r) = +1$ or $\chi(a_1, \ldots, a_r) = -1$. Since we only consider general position, we do not allow $\chi(a_1, \ldots, a_r) = 0$ for distinct points $a_1, \ldots, a_r$.

The chirotope χ fulfills the conditions from Theorem 5 which we can encode with $\Theta(n^{r+2})$ clauses similar as in the planar case [41]: The first property is clearly fulfilled because of the general position. For the second property, we have to ensure that $\chi(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = \operatorname{sgn}(\sigma) \cdot \chi(a_1, \ldots, a_r)$ holds for every r distinct indices $a_1, \ldots, a_r \in \{1, \ldots, n\}$ and every permutation σ on $\{1, \ldots, r\}$. Depending on σ, this translates into an equation of the form $A = B$ or $A = \neg B$ where $A, B \in \{true, false\}$ are Boolean variables. This further translates into constraints of the form $(\neg A \vee B) \wedge (A \vee \neg B)$ or $(\neg A \vee \neg B) \wedge (A \vee B)$, respectively. For the third property, we have to deal with an implication of the form

$$(A \cdot B \geq 0) \wedge (C \cdot D \geq 0) \Longrightarrow (E \cdot F \geq 0), \tag{1}$$

where $A, B, C, D, E, F \in \{+1, 0, -1\}$ capture the orientation of an r-tuple of not necessarily distinct points. Since $A \cdot B \geq 0$ (and analogously $C \cdot D \geq 0$) is fulfilled in the seven cases

$$(A, B) \in \{(+1, +1), (+1, 0), (0, +1), (0, 0), (0, -1), (-1, 0), (-1, -1)\},$$

and $E \cdot F \geq 0$ is not fulfilled in the two cases

$$(E, F) \in \{(+1, -1), (-1, +1)\},$$

the implication (1) is violated in $7 \cdot 7 \cdot 2 = 98$ of $3^6 = 729$ cases. We can forbid each of these cases and ensure that implication (1) holds by the following 98 constraints:

$$\begin{array}{l}
A \neq +1 \ \vee \ B \neq +1 \ \vee \ C \neq +1 \ \vee \ D \neq +1 \ \vee \ E \neq +1 \ \vee \ F \neq -1, \\
A \neq +1 \ \vee \ B \neq +1 \ \vee \ C \neq +1 \ \vee \ D \neq +1 \ \vee \ E \neq -1 \ \vee \ F \neq +1, \\
A \neq +1 \ \vee \ B \neq +1 \ \vee \ C \neq +1 \ \vee \ D \neq \ \ 0 \ \vee \ E \neq +1 \ \vee \ F \neq -1, \\
A \neq +1 \ \vee \ B \neq +1 \ \vee \ C \neq +1 \ \vee \ D \neq \ \ 0 \ \vee \ E \neq -1 \ \vee \ F \neq +1, \\
\qquad\qquad\qquad\qquad\qquad \vdots \\
A \neq -1 \ \vee \ B \neq -1 \ \vee \ C \neq -1 \ \vee \ D \neq -1 \ \vee \ E \neq +1 \ \vee \ F \neq -1, \\
A \neq -1 \ \vee \ B \neq -1 \ \vee \ C \neq -1 \ \vee \ D \neq -1 \ \vee \ E \neq -1 \ \vee \ F \neq +1.
\end{array}$$

Whenever A captures the orientation of an r-tuple with duplicate points, we omit the term $A \neq 0$ because it cannot be satisfied. And whenever A captures the orientation of r distinct points $a_1, \ldots, a_r$, $A \neq 0$ is trivially fulfilled because of the general position, and we can replace $A \neq -1$ by $X_{a_1,\ldots,a_r}$ and $A \neq +1$ by $\neg X_{a_1,\ldots,a_r}$. Analogous arguments apply to B, C, D, E, F. This completes the encoding of the chirotope.

To ensure that the chirotope is acyclic, we introduce constraints as discussed in Section 3.1: For every r distinct points $a_1, \ldots, a_r$, we ensure that $\chi(a_1, a_2, \ldots, a_r) = -$ or $\chi(a_1, \ldots, a_{i-1}, a_{r+1}, a_{i+1}, \ldots, a_r) = +$ holds for some $i = 1, \ldots, r$ via the constraint

$$\neg X_{a_1,a_2,\ldots,a_r} \vee \bigvee_{i=1}^{r} X_{a_1,\ldots,a_{i-1},a_{r+1},a_{i+1},\ldots,a_r}.$$

Next, we introduce for every $r+1$ distinct points $a_1,\dots,a_{r+1}$ an auxiliary variable $S_{a_1,\dots,a_{r-1};a_r,a_{r+1}}$ to indicate whether the hyperplane determined by $\{p_{a_1},\dots,p_{a_{r-1}}\}$ separates the two points p_{a_r} and $p_{a_{r+1}}$, and an auxiliary variable $C_{a_1,\dots,a_r;a_{r+1}}$ whether the point $p_{a_{r+1}}$ is contained in the simplex spanned by $\{p_{a_1},\dots,p_{a_r}\}$. As discussed in Section 3.1, the values of these auxiliary variables are fully determined by the chirotope variables:

$$S_{a_1,\dots,a_{r-1};a_r,a_{r+1}} :\Leftrightarrow X_{a_1,\dots,a_{r-1},a_r} \neq X_{a_1,\dots,a_{r-1},a_{r+1}}$$

and

$$\begin{aligned} C_{a_1,\dots,a_r;a_{r+1}} &:\Leftrightarrow X_{a_1,\dots,a_{i-1},a_i,a_{i+1},\dots,a_r} = X_{a_1,\dots,a_{i-1},a_{r+1},a_{i+1},\dots,a_r} \text{ for all } i=1,\dots,r \\ &\Leftrightarrow \bigvee_{i=1}^{r} \neg S_{a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;a_i,a_{r+1}}. \end{aligned}$$

Using these auxiliary variables we can formulate $\binom{n}{k}$ clauses to assert that there are no k-gons in S: Among every subset $T \subset S$ of size $|T| = k$ there is at least one point $p \in T$ which is contained in a simplex spanned by r points of $T \setminus \{p\}$:

$$\bigvee_{a_1,\dots,a_{r+1} \in T \text{ distinct}} C_{a_1,\dots,a_r;a_{r+1}}$$

To assert that there are no k-holes in S, we can proceed in a similar manner: For every subset $T \subset S$ of size $|T| = k$ there is at least one point $p \in S$ which is contained in a simplex spanned by r points of $T \setminus \{p\}$:[4]

$$\bigvee_{a_1,\dots,a_r \in T \text{ and } a_{r+1} \in S \text{ distinct}} C_{a_1,\dots,a_r;a_{r+1}}$$

Altogether, we created a Boolean satisfiability instance that is satisfiable if and only if there exists an acyclic rank $r = d+1$ chirotope on n elements without k-gons or k-holes, respectively. Moreover, since all constraints are necessary and sufficient for the desired properties, and all auxiliary variables are fully determined by the X-variables (which encode the chirotope), the solutions of the SAT instance are in bijection with acyclic chirotopes of rank $r = d+1$ on n elements without k-gons or k-holes, respectively. And in particular, if the instance is unsatisfiable, no such chirotope (and hence no point set) exists, and we can conclude $g^{(d)}(k) \leq n$ or $h^{(d)}(k) \leq n$, respectively.

5 Computer-assisted proofs

To determine satisfiability of the generated SAT instances, we use the SAT solver `CaDiCaL`, version 1.0.3 [5]. If an instance is unsatisfiable, `CaDiCaL` generates a DRAT certificate which can then be verified by an independent proof checking tool such as `DRAT-trim` [49]. All our computations were performed on a single CPU. Since some of the computations required a lot of resources (up to 300 GB of RAM and disk space), we used a node with 1 TB of RAM from the computing cluster of the Institute of Mathematics at TU Berlin. The python programs for creating the instances and further information is available as supplemental data [39].

[4] Moreover observe that, since the convex hull of $T \setminus \{p\}$ can be triangulated in a way such that all simplices of the triangulation contain the leftmost point of $T \setminus \{p\}$, the point p lies in a simplex determined by the leftmost and d further points from $T \setminus \{p\}$. However, even though this observation can be used to slightly reduce the size of an instance, `CaDiCaL` performed worse in all our experiments.

5.1 Resources for proving Theorem 2

To verify Theorem 2 we used the following resources.

- $g^{(3)}(7) \leq 13$: The size of the instance is about 245 MB and `CaDiCaL` with parameter `--unsat` managed to prove unsatisfiability in about 2 CPU days. The created DRAT certificate is about 35 GB and the verification with `DRAT-trim` took about 1 CPU day.[5]
- $h^{(3)}(7) \leq 14$: The size of the instance is about 433 MB and `CaDiCaL` with parameter `--unsat` managed to prove unsatisfiability in about 11 CPU days. The created DRAT certificate is about 196 GB and the verification with `DRAT-trim` took about 10 CPU days.[6]
- $h^{(4)}(8) \leq 13$: The size of the instance is about 955 MB and `CaDiCaL` with parameter `--unsat` managed to prove unsatisfiability in about 7 CPU days. The created DRAT certificate is about 163 GB and the verification with `DRAT-trim` took about 5 CPU days.[7]
- $h^{(5)}(9) \leq 13$: The size of the instance is about 4.2 GB and `CaDiCaL` (without parameter `--unsat`) managed to prove unsatisfiability in about 2 CPU days. The created DRAT certificate is about 95 GB and the verification with `DRAT-trim` took about 2 CPU days.[8]

5.2 Verification of previous results

We have also used our SAT framework to verify the previously known bounds. For the unsatisfiability of $g^{(2)}(6) \leq 17$ [44], `CaDiCaL` takes about 10 CPU minutes to create a DRAT certificate. The certificate has about 668 MB and can be verified by `DRAT-trim` within 12 CPU minutes. The instances $g^{(2)}(5) \leq 9$ (Makai 1935), $h^{(2)}(5) \leq 10$ [24], and $h^{(3)}(6) \leq 9$ [7] can be solved and verified within only a few CPU seconds and the respective DRAT certificates have only a few MB.

It is worth noting that the solver `CaDiCaL` performs significantly faster on the instance for $g^{(2)}(6) \leq 17$ than the solver `glucose` [4] which we used in [41] (see also [34]) and that `CaDiCaL` and `DRAT-trim` perform about twice as fast on our "old" instance for $g^{(2)}(6) \leq 17$ from [41], which is based on signotopes[9].

5.3 Existence of planar 6-holes

Last but not least, we present a computer-assisted proof for the existence of 6-holes in planar point sets.

Proof of Theorem 3. Suppose towards a contradiction that there exists a set S of $g^{(2)}(9)$ points in the plane which does not contain a 6-hole. By definition of $g^{(2)}(9)$, there exists a subset $A \subseteq S$ of $|A| = 9$ points which determine a 9-gon. We now restrict our attention to $S' = S \cap \operatorname{conv}(A)$.

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5 We also ran `CaDiCaL` with default parameters (without parameter `--unsat`) for the instance $g^{(3)}(7) \leq 13$. The computations took about 2 CPU days, the DRAT certificate is about 43 GB, and verification with `DRAT-trim` took about 2 CPU days.

6 We also ran `CaDiCaL` with default parameters (without parameter `--unsat`) for the instance $h^{(3)}(7) \leq 14$. The computations took about 14 CPU days, the DRAT certificate is about 282 GB, and verification with `DRAT-trim` took about 12 CPU days.

7 We also ran `CaDiCaL` with default parameters (without parameter `--unsat`) for the instance $h^{(4)}(8) \leq 13$. The computations took about 8 CPU days, the DRAT certificate is about 315 GB, and verification with `DRAT-trim` took about 4 CPU days.

8 We also ran `CaDiCaL` with parameter `--unsat` for the instance $h^{(5)}(9) \leq 13$. The computations took about 4 CPU days, the DRAT certificate is about 105 GB, and verification with `DRAT-trim` took about 2 CPU days.

9 While planar point sets can be modeled as signotopes of rank 3, this is not true in higher dimensions.

Let B denote the extremal points of $S' \cap \mathrm{int}(\mathrm{conv}(A))$, let C denote the extremal points of $S' \cap \mathrm{int}(\mathrm{conv}(B))$, and let $D = S' \cap \mathrm{int}(\mathrm{conv}(C))$. Here $\mathrm{int}(X)$ denotes the interior of X. In other words, A, B, and C span the first, second, and third convex hull layer of $\mathrm{conv}(S')$, respectively, and D contains all points that lie on or inside the fourth convex hull layer.

Lemma 1 from [48] (and also Theorem 4 from [37]) now asserts that $D = \emptyset$, as otherwise S' (and thus S) would contain a 6-hole. Since the points of C form a $|C|$-hole, it clearly holds $|C| \le 5$. Furthermore, we may assume that A is a *minimal 9-gon in* S', that is, A is the only 9-gon in S', as otherwise we could iteratively choose 9-gons inside S' which contain fewer 9-gons until we end up with a minimal one. Hence, we have $|B| \le 8$, and in total

$$|S'| \le |A| + |B| + |C| \le 9 + 8 + 5 = 22.$$

By slightly modifying our framework from Section 4, we can assert that the 9 points of A form a 9-gon and all other points of $S' \setminus A$ lie in the interior of $\mathrm{conv}(A)$. Using `CaDiCaL` we managed to verify that, for every $n = 9, \ldots, 22$, there exists no set S' with $|S'| = n$ that has the desired properties. This contradiction shows that every a set of $g^{(2)}(9)$ points in the plane contains a 6-hole. ◀

The total size of all 14 instances in the proof of Theorem 3 is about 3.8 GB. The total computation time of `CaDiCaL` and `DRAT-trim` to create and verify the DRAT certificates for the instances is about 1 CPU hour. The total size of the generated DRAT certificates is about 6.2 GB.

6 Discussion

We have presented a SAT framework to investigate problems on acyclic chirotopes of small ranks. Using modern SAT solvers we obtained new results for Erdős–Szekeres-type problems in dimensions $d = 3, 4, 5$, and we verified various previously known results. In particular, we have verified Gerken's proof for the existence of 6-holes in planar point sets. Moreover, acyclic chirotopes on $n = 29$ points without 6-holes were found within a few minutes, but for $n = 30$ the computations did not terminate within months. Based on this computation evidence we conjecture that every set of 30 points in the plane contains at least one 6-hole, that is, $h^{(2)}(6) = 30$.

We have also used our framework to investigate the existence of 8-gons and 8-holes in 3-space. `CaDiCaL` managed to find a rank 4 chirotope on 17 elements without 8-gons and a rank 4 chirotope on 18 elements without 8-holes; see the supplemental data [39]. The computations took 100 CPU days and 24 CPU days, respectively. Since it gets harder to find chirotopes without 8-holes as the number of points increases (8-hole-free chirotopes on 17 or less elements can be found within only a few CPU hours), this is computational evidence that sufficiently large sets in 3-space contain 8-holes.

Since some of the unsatisfiability certificates generated in course of this article grew very large, it would be interesting to further optimize the SAT model by breaking further symmetries so that the solver can terminate faster and the obtained DRAT certificates become smaller.

Last but not least we want to mention that our framework can also be used to tackle other problems on higher dimensional point sets or oriented matroids. By slightly adapting our model, we managed to answer a Tverberg-type question by Fulek et al. (see Section 3.2 in [22] for more details) and we managed to find a chirotope which is a contact representation of a particular hypergraph, partially answering a question by Evans et al. [17].

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